Chapter - Continuity & Differentiability

Topic-1: Continuity

1 MCQs with One Correct Answer

1. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at [1999 - 2 Marks]
   (a) all integers
   (b) all integers except 0 and 1
   (c) all integers except 0
   (d) all integers except 1

2. The function $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, [.] denotes the greatest integer function, is discontinuous at [1995S]
   (a) All x (b) All integer points
   (c) No x (d) x which is not an integer

4 Fill in the Blanks

3. Let $f(x) = [x] \sin\left(\frac{\pi}{[x+1]}\right)$, where [•] denotes the greatest integer function. The domain of f is... and the points of discontinuity of f in the domain are..... (1996 - 2 Marks)

4. Let $f(x) = \begin{cases} \frac{(x^3+x^2-16x+20)}{(x-2)^2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$

   If $f(x)$ is continuous for all x, then $k = ...$ [1981 - 2 Marks]

6 MCQs with One or More Than One Correct

5. Let [x] be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x+[x]))$ is discontinuous? [Adv. 2017]
   (a) $x = -1$ (b) $x = 0$
   (c) $x = 1$ (d) $x = 2$

6. For every pair of continuous functions $f, g : [0, 1] \to R$ such that $\max\{f(x) : x \in [0,1]\} = \max\{g(x) : x \in [0,1]\}$, the correct statement(s) is (are): [Adv. 2014]
   (a) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
   (b) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
   (c) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$
   (d) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

7. For every integer n, let a_n and b_n be real numbers. Let function $f : IR \to IR$ be given by [2012]

   $$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases}$$

   for all integers n. If f is continuous, then which of the following hold(s) for all n?
   (a) $a_{n-1} - b_{n-1} = 0$ (b) $a_n - b_n = 1$
   (c) $a_n - b_{n+1} = 1$ (d) $a_{n-1} - b_n = -1$

8. The following functions are continuous on $(0, \pi)$. [1991 - 2 Marks]
   (a) $\tan x$
   (b) $\int_0^x t \sin\frac{1}{t}\, dt$
   (c) $\begin{cases} 1, & 0 < x \le \frac{3\pi}{4} \\ 2\sin\frac{2}{9}x, & \frac{3\pi}{4} < x < \pi \end{cases}$
   (d) $\begin{cases} x\sin x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

9. If $f(x) = \frac{1}{2}x - 1$, then on the interval $[0, \pi]$ [1989 - 2 Marks]
(a) $\tan[f(x)]$ and $1/f(x)$ are both continuous
(b) $\tan[f(x)]$ and $1/f(x)$ are both discontinuous
(c) $\tan[f(x)]$ and $f^{-1}(x)$ are both continuous
(d) $\tan[f(x)]$ is continuous but $1/f(x)$ is not.

10 Subjective Problems

10. Let $f(x) = \begin{cases} \{1+|\sin x|\}^{a/|\sin x|} & ; \quad \frac{-\pi}{6} < x < 0 \\ b & ; \quad x = 0 \\ e^{\tan 2x/\tan 3x} & ; \quad 0 < x < \frac{\pi}{6} \end{cases}$

[1994 - 4 Marks]

Determine a and b such that $f(x)$ is continuous at $x = 0$

11. Let $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x > 0 \end{cases}$ [1990 - 4 Marks]

Determine the value of a, if possible, so that the function is continuous at $x = 0$

12. Find the values of a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b. & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x. & \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$. [1989 - 2 Marks]

13. Let $f(x)$ be a continuous and $g(x)$ be a discontinuous function. prove that $f(x) + g(x)$ is a discontinuous function. [1987 - 2 Marks]

14. Let $f(x) = \begin{cases} 1+x, 0 \le x < 2 \\ 3-x, 2 \le x \le 3 \end{cases}$ [1983 - 2 Marks]

Determine the form of $g(x) = f(f(x))$ and hence find the points of discontinuity of g, if any

15. Let $f(x+y) = f(x) + f(y)$ for all x and y. If the function $f(x)$ is continuous at $x = 0$, then show that $f(x)$ is continuous at all x. [1981 - 2 Marks]

Topic-2: Differentiability

1 MCQs with One Correct Answer

1. Let $f(x)$ be a continuously differentiable function on the interval $(0, \infty)$ such that $f(1) = 2$ and $\lim_{t \to x} \frac{t^{10} f(x) - x^{10} f(t)}{t^9 - x^9} = 1$ for each $x > 0$. Then, for all $x > 0$, $f(x)$ is equal to [Adv. 2024]
(a) $\frac{31}{11x} - \frac{9}{11}x^{10}$
(b) $\frac{9}{11x} + \frac{13}{11}x^{10}$
(c) $\frac{-9}{11x} + \frac{31}{11}x^{10}$
(d) $\frac{13}{11x} + \frac{9}{11}x^{10}$

2. Let $f(x) = \begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & x \ne 0 \\ 0, & x = 0 \end{cases}, x \in R$ then f is [2012]
(a) differentiable both at $x = 0$ and at $x = 2$
(b) differentiable at $x = 0$ but not differentiable at $x = 2$
(c) not differentiable at $x = 0$ but differentiable at $x = 2$
(d) differentiable neither at $x = 0$ nor at $x = 2$

3. Let $g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}$; $0 < x < 2$, m and n are integers, $m \ne 0$, $n > 0$, and let p be the left hand derivative of $|x-1|$ at $x = 1$. If $\lim_{x \to 1^+} g(x) = p$, then [2008]
(a) $n = 1, m = 1$
(b) $n = 1, m = -1$
(c) $n = 2, m = 2$
(d) $n > 2, m = n$

4. Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$, and $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is [2007 - 3 marks]
(a) $\frac{1}{3x} + \frac{2x^2}{3}$
(b) $\frac{-1}{3x} + \frac{4x^2}{3}$
(c) $\frac{-1}{x} + \frac{2}{x^2}$
(d) $\frac{1}{x}$

5. If $f(x)$ is continuous and differentiable function and $f(1/n)=0\ \forall\ n\geq 1$ and $n\in I$, then [2005S]
(a) $f(x)=0, x\in(0,1]$
(b) $f(0)=0, f'(0)=0$
(c) $f(0)=0=f'(0),\ x\in(0,1]$
(d) $f(0)=0$ and $f'(0)$ need not to be zero

6. The function given by $y=||x|-1|$ is differentiable for all real numbers except the points [2005S]
(a) $\{0,1,-1\}$ (b) ±1 (c) 1 (d) -1

7. The domain of the derivative of the function

$$f(x)=\begin{cases}\tan^{-1}x & \text{if } |x|\leq 1\\ \frac{1}{2}(|x|-1) & \text{if } |x|>1\end{cases}$$ is [2002S]

(a) $R-\{0\}$ (b) $R-\{1\}$
(c) $R-\{-1\}$ (d) $R-\{-1,1\}$

8. Which of the following functions is differentiable at $x=0$?
(a) $\cos(|x|)+|x|$ (b) $\cos(|x|)-|x|$ [2001S]
(c) $\sin(|x|)+|x|$ (d) $\sin(|x|)-|x|$

9. Let $f:R\to R$ be a function defined by $f(x)=\max\{x,x^3\}$. The set of all points where $f(x)$ is NOT differentiable is [2001S]
(a) $\{-1,1\}$ (b) $\{-1,0\}$ (c) $\{0,1\}$ (d) $\{-1,0,1\}$

10. The left-hand derivative of $f(x)=[x]\sin(\pi x)$ at $x=k$, k an integer, is [2001S]
(a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1}k\pi$

11. The function $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is NOT differentiable at [1999 - 2 Marks]
(a) -1 (b) 0 (c) 1 (d) 2

12. Let [.] denote the greatest integer function and $f(x)=[\tan^2 x]$, then: [1993 - 1 Mark]
(a) $\lim_{x\to0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x=0$
(c) $f(x)$ is not differentiable at $x=0$
(d) $f'(0)=1$

13. Let $f:R\to R$ be a differentiable function and $f(1)=4$. Then the value of $\lim_{x\to1}\int_4^{f(x)}\frac{2t}{x-1}dt$ is [1990 - 2 Marks]
(a) $8f'(1)$ (b) $4f'(1)$ (c) $2f'(1)$ (d) $f'(1)$

14. If $f(a)=2,\ f'(a)=1,\ g(a)=-1,\ g'(a)=2$, then the value of $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$ is [1983 - 1 Mark]
(a) -5 (b) $\frac{1}{5}$
(c) 5 (d) none of these

15. For a real number y, let $[y]$ denotes the greatest integer less than or equal to y: Then the function $f(x)=\frac{\tan(\pi[x-\pi])}{1+[x]^2}$ is [1981 - 2 Marks]
(a) discontinuous at some x
(b) continuous at all x, but the derivative $f'(x)$ does not exist for some x
(c) $f'(x)$ exists for all x, but the second derivative $f''(x)$ does not exist for some x
(d) $f'(x)$ exists for all x

2 Integer Value Answer/ Non-Negative Integer

16. Let the functions $f:(-1,1)\to\mathbb{R}$ and $g:(-1,1)\to(-1,1)$ be defined by
$f(x)=|2x-1|+|2x+1|$ and $g(x)=x-[x]$,
where $[x]$ denotes the greatest integer less than or equal to x. Let $f\circ g:(-1,1)\to\mathbb{R}$ be the composite function defined by $(f\circ g)(x)=f(g(x))$. Suppose c is the number of points in the interval $(-1,1)$ at which $f\circ g$ is NOT continuous, and suppose d is the number of points in the interval $(-1,1)$ at which $f\circ g$ is NOT differentiable. Then the value of $c+d$ is ______ [Adv. 2020]

17. Let $f:R\to R$ and $g:R\to R$ be respectively given by $f(x)=|x|+1$ and $g(x)=x^2+1$. Define $h:R\to R$ by
$$h(x)=\begin{cases}\max\{f(x),g(x)\} & \text{if } x\leq 0,\\ \min\{f(x),g(x)\} & \text{if } x>0.\end{cases}$$
The number of points at which $h(x)$ is not differentiable is [Adv. 2014]

18. Let $f:[1,\infty)\to[2,\infty)$ be a differentiable function such that $f(1)=2$. If $6\int_1^x f(t)dt=3xf(x)-x^3$ for all $x\geq1$, then the value of $f(2)$ is [2011]

4 Fill in the Blanks

19. Let $f(x)=x|x|$. The set of points where $f(x)$ is twice differentiable is .... [1992 - 2 Marks]

20. Let $f(x)=\begin{cases}(x-1)^2\sin\frac{1}{(x-1)}-|x| & \text{if } x\neq1\\ -1, & \text{if } x=1\end{cases}$

be a real-valued function. Then the set of points where $f(x)$ is not differentiable is .... [1981 - 2 Marks]

6 MCQs with One or More Than One Correct

21. Let $S=(0,1)\cup(1,2)\cup(3,4)$ and $T=\{0,1,2,3\}$. Then which of the following statements is(are) true? [Adv. 2023]
(a) There are infinitely many functions from S to T
(b) There are infinitely many strictly increasing functions from S to T
(c) The number of continuous functions from S to T is at most 120
(d) Every continuous function from S to T is differentiable

22. Let $f:(0,1)\to\mathbb{R}$ be the function defined as $f(x)=[4x]\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to x. Then which of the following statements is(are) true? [Adv. 2023]
(a) The function f is discontinuous exactly at one point in (0,1)
(b) There is exactly one point in (0,1) at which the function f is continuous but NOT differentiable
(c) The function f is NOT differentiable at more than three points in (0,1)
(d) The minimum value of the function f is $-\frac{1}{512}$

23. Let the function $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^3-x^2+(x-1)\sin x$ and let $g:\mathbb{R}\to\mathbb{R}$ be an arbitrary function. Let $fg:\mathbb{R}\to\mathbb{R}$ be the product function defined by $(fg)(x)=f(x)g(x)$. Then which of the following statements is/are TRUE? [Adv. 2020]
(a) If g is continuous at $x=1$, then fg is differentiable at $x=1$
(b) If fg is differentiable at $x=1$, then g is continuous at $x=1$
(c) If g is differentiable at $x=1$, then fg is differentiable at $x=1$
(d) If fg is differentiable at $x=1$, then g is differentiable at $x=1$

24. Let $f:R\to R$ given by [Adv. 2019]

$$f(x)=\begin{cases}x^5+5x^4+10x^3+10x^2+3x+1, & x<0;\\ x^2-x+1, & 0\le x<1;\\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3}, & 1\le x<3;\\ (x-2)\log_e(x-2)-x+\frac{10}{3}, & x\ge 3\end{cases}$$

Then which of the following options is/are correct ?
(a) f' has a local maximum at $x=1$
(b) f is increasing on $(-\infty,0)$
(c) f' is NOT differentiable at $x=1$
(d) f is onto

25. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two non-constant differentiable functions. If

$$f'(x)=(e^{(f(x)-g(x))})g'(x) \text{ for all } x\in\mathbb{R},$$

and $f(1)=g(2)=1$, then which of the following statement (s) is (are) TRUE ? [Adv. 2018]
(a) $f(2)<1-\log_e 2$
(b) $f(2)>1-\log_e 2$
(c) $g(1)>1-\log_e 2$
(d) $g(1)<1-\log_e 2$

26. Let $f:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ and $g:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ be functions defined by $f(x)=[x^2-3]$ and $g(x)=|x|f(x)+|4x-7|f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y\in R$. Then [Adv. 2016]
(a) f is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$
(b) f is discontinuous exactly at four points in $\left[-\frac{1}{2},2\right]$
(c) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2},2\right)$
(d) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2},2\right)$

27. Let $a,b\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=a\cos(|x^3-x|)+b|x|\sin(|x^3+x|)$.
Then f is [Adv. 2016]
(a) differentiable at x=0 if a=0 and b=1
(b) differentiable at x=1 if a=1 and b=0
(c) NOT differentiable at x=0 if a=1 b=0
(d) NOT differentiable at x=1 if a=1 and b=1

28. Let $g:R\to R$ be a differentiable function with $g(0)=0$, $g'(0)=0$ and $g'(1)\neq 0$. Let $f(x)=\begin{cases}\frac{x}{|x|}g(x), & x\neq 0\\ 0, & x=0\end{cases}$ and $h(x)=e^{|x|}$ for all $x\in R$. Let $(foh)(x)$ denote $f(h(x))$ and $(hof)(x)$ denote $h(f(x))$. Then which of the following is (are) true? [Adv. 2015]
(a) f is differentiable at $x=0$
(b) h is differentiable at $x=0$
(c) foh is differentiable at $x=0$
(d) hof is differentiable at $x=0$

29. If $f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\le-\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \ell nx, & x>1\end{cases}$, then [2011]

(a) $f(x)$ is continuous at $x=-\frac{\pi}{2}$
(b) $f(x)$ is not differentiable at $x=0$
(c) $f(x)$ is differentiable at $x=1$
(d) $f(x)$ is differentiable at $x=-\frac{3}{2}$

30. Let $f: R\to R$ be a function such that $f(x+y)=f(x)+f(y)$, $\forall\ x, y\in R$. If $f(x)$ is differentiable at $x=0$, then [2011]
(a) $f(x)$ is differentiable only in a finite interval containing zero
(b) $f(x)$ is continuous $\forall\ x\in R$
(c) $f'(x)$ is constant $\forall\ x\in R$
(d) $f(x)$ is differentiable except at finitely many points.

31. If $f(x)=\min\{1,x^2,x^3\}$, then [2006 - 5M, –1]
(a) $f(x)$ is continuous $\forall\ x\in R$
(b) $f(x)$ is continuous and differentiable everywhere.
(c) $f(x)$ is not differentiable at two points
(d) $f(x)$ is not differentiable at one point

32. Let $h(x)=\min\{x,x^2\}$, for every real number of x, Then [1998 - 2 Marks]
(a) h is continuous for all x
(b) h is differentiable for all x
(c) $h'(x)=1$, for all $x>1$
(d) h is not differentiable at two values of x.

33. The function $f(x)=\max\{(1-x),(1+x),2\},\ x\in(-\infty,\infty)$ is [1995]
(a) continuous at all points
(b) differentiable at all points
(c) differentiable at all points except at $x=1$ and $x=-1$
(d) continuous at all points except at $x=1$ and $x=-1$, where it is discontinuous

34. Let $g(x)=x f(x)$, where $f(x)=\begin{cases}x\sin\frac{1}{x}, & x\ne 0\\ 0, & x=0\end{cases}$. At $x=0$
(a) g is differentiable but g' is not continuous [1994]
(b) g is differentiable while f is not
(c) both f and g are differentiable
(d) g is differentiable and g' is continuous

35. Let $f(x)=\begin{cases}0, & x<0\\ x^2, & x\ge 0\end{cases}$ then for all x [1994]
(a) f' is differentiable (b) f is differentiable
(c) f' is continuous (d) f is continuous

36. The function $f(x)=\begin{cases}|x-3|, & x\ge 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1\end{cases}$ is [1988 - 2 Marks]
(a) continous at $x=1$ (b) differentiable at $x=1$
(c) continous at $x=3$ (d) differentiable at $x=3$.

37. The set of all points where the function $f(x)=\frac{x}{(1+|x|)}$ is differentiable, is [1987 - 2 Marks]
(a) $(-\infty,\infty)$ (b) $[0,\infty)$
(c) $(-\infty,0)\cup(0,\infty)$ (d) $(0,\infty)$
(e) None

38. Let $[x]$ denote the greatest integer less than or equal to x. If $f(x)=[x\sin\pi x]$, then $f(x)$ is [1986 - 2 Marks]
(a) continuous at $x=0$ (b) continuous in $(-1,0)$
(c) differentiable at $x=1$ (d) differentiable in $(-1,1)$
(e) none of these

39. The function $f(x)=1+|\sin x|$ is [1986 - 2 Marks]
(a) continuous nowhere
(b) continuous everywhere
(c) differentiable nowhere
(d) not differentiable at $x=0$
(e) not differentiable at infinite number of points.

40. If $f(x)=x(\sqrt{x}-\sqrt{x+1})$, then– [1985 - 2 Marks]
(a) $f(x)$ is continuous but not differentiable at $x=0$
(b) $f(x)$ is differentiable at $x=0$
(c) $f(x)$ is not differentiable at $x=0$
(d) none of these

41. If $x+|y|=2y$, then y as a function of x is [1984 - 3 Marks]
(a) defined for all real x
(b) continuous at $x=0$
(c) differentiable for all x
(d) such that $\frac{dy}{dx}=\frac{1}{3}$ for $x<0$

7 Match the Following

42. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be functions defined by

$$f(x)=\begin{cases}x|x|\sin\left(\frac{1}{x}\right), & x\ne 0\\ 0, & x=0\end{cases}$$ and

$$g(x)=\begin{cases}1-2x, & 0\le x\le\frac{1}{2}\\ 0, & \text{otherwise.}\end{cases}$$

Let $a, b, c, d\in\mathbb{R}$. Define the function $h:\mathbb{R}\to\mathbb{R}$ by

$$h(x) = a\,f(x) + b\left(g(x) + g\left(\frac{1}{2} - x\right)\right) + c(x - g(x)) + d\,g(x), x \in \mathbb{R}$$

Match each entry in List-I to the correct entry in List-II.

List-I	List-II
(P) If $a = 0, b = 1, c = 0$, and $d = 0$, then	(1) h is one-one.
(Q) If $a = 1, b = 0, c = 0$, and $d = 0$, then	(2) h is onto.
(R) If $a = 0, b = 0, c = 1$, and $d = 0$, then	(3) h is differentiable on R
(S) If $a = 0, b = 0, c = 0$, and $d = 1$ then	(4) the range of h is [0, 1]
	(5) the range of h is $\{0,1\}$.

The correct option is **[Adv. 2024]**

(a) (P)→(4) (Q)→(3) (R)→(1) (S)→(2)
(b) (P)→(5) (Q)→(2) (R)→(4) (S)→(3)
(c) (P)→(5) (Q)→(3) (R)→(2) (S)→(4)
(d) (P)→(4) (Q)→(2) (R)→(1) (S)→(3)

43. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined by **[Adv. 2018]**

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$,

(ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x+2))]$, where, for $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

LIST - I	LIST - II
P. The function f_1 is	1. NOT continuous at $x = 0$
Q. The function f_2 is	2. continuous at $x = 0$ and NOT differentiable at $x = 0$
R. The function f_3 is	3. differentiable at $x = 0$ and its derivative is NOT continuous at $x = 0$
S. The function f_4 is	4. differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is:

(a) P→2; Q→3; R→1; S→4
(b) P→4; Q→1; R→2; S→3
(c) P→4; Q→2; R→1; S→3
(d) P→2; Q→1; R→4; S→3

44. Let $f_1 : R \to R$, $f_2 : [0, \infty) \to R$, $f_3 : R \to R$ and $f_4 : R \to [0, \infty)$ be defined by $f_1(x) = \begin{cases} |x| & \text{if } x < 0, \\ e^x & \text{if } x \geq 0; \end{cases}$

$f_2(x) = x^2$; $f_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x & \text{if } x \geq 0; \end{cases}$ and $f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0, \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0. \end{cases}$ **[Adv. 2014]**

List-I	List-II
P. f_4 is	1. Onto but not one-one
Q. f_3 is	2. Neither continuous nor one-one
R. $f_2 o f_1$ is	3. Differentiable but not one-one
S. f_2 is	4. Continuous and one-one

(a) P→3; Q→1; R→4; S→2
(b) P→1; Q→3; R→4; S→2
(c) P→3; Q→1; R→2; S→4
(d) P→1; Q→3; R→2; S→4

45. In the following $[x]$ denotes the greatest integer less than or equal to x.

Match the functions in Column I with the properties in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS. **[2007 - 6 marks]**

Column I	Column II
(A) $x\|x\|$	(p) continuous in $(-1, 1)$
(B) $\sqrt{\|x\|}$	(q) differentiable in $(-1, 1)$
(C) $x+[x]$	(r) strictly increasing in $(-1, 1)$
(D) $\|x-1\|+\|x+1\|$	(s) not differentiable at least at one point in $(-1, 1)$

46. In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book. **[1992 - 2 Marks]**

Column I	Column II
(A) $\sin(\pi[x])$	(p) differentiable everywhere
(B) $\sin(\pi(x-[x])$	(q) nowhere differentiable
	(r) not differentiable at 1 and – 1

10 Subjective Problems

47. If $f(x-y)=f(x)\cdot g(y)-f(y)\cdot g(x)$ and $g(x-y)=g(x)\cdot g(y)-f(x)\cdot f(y)$ for all $x, y \in R$. If right hand derivative at $x=0$ exists for $f(x)$. Find derivative of $g(x)$ at $x=0$ **[2005 - 4 Marks]**

48. If $|c| \le \frac{1}{2}$ and $f(x)$ is a differentiable function at $x=0$ given

by $f(x)=\begin{cases} b\sin^{-1}\left(\frac{c+x}{2}\right) & , -\frac{1}{2}<x<0 \\ \frac{1}{2} & , x=0 \\ \frac{e^{ax/2}-1}{x} & , 0<x<\frac{1}{2} \end{cases}$.

Find the value of 'a' and prove that $64b^2=4-c^2$ **[2004 - 4 Marks]**

49. If a function $f:[-2a, 2a]\to R$ is an odd function such that $f(x)=f(2a-x)$ for $x\in[a, 2a]$ and the left hand derivative at $x=a$ is 0 then find the left hand derivative at $x=-a$. **[2003 - 2 Marks]**

50. Let $f(x)=\begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x\ge 0, \end{cases}$ and **[2002 - 5 Marks]**

$g(x)=\begin{cases} x+1 & \text{if } x<0 \\ (x-1)^2+b & \text{if } x\ge 0, \end{cases}$ where a and b are non-negative real numbers. Determine the composite function $g\,o\,f$. If $(g\,o\,f)(x)$ is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is g o f differentiable at $x=0$? Justify your answer.

51. Let $\alpha\in R$. Prove that a function $f: R\to R$ is differentiable at α if and only if there is a function $g: R\to R$ which is continuous at α and satisfies $f(x)-f(\alpha)=g(x)(x-\alpha)$ for all $x\in R$. **[2001 - 5 Marks]**

52. Determine the values of x for which the following function fails to be continuous or differentiable: **[1997 - 5 Marks]**

$f(x)=\begin{cases} 1-x, & x<1 \\ (1-x)(2-x), & 1\le x\le 2 \\ 3-x, & x>2 \end{cases}$ Justify your answer.

53. Let $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ for all real x and y. If $f'(0)$ exists and equals -1 and $f(0)=1$, find $f(2)$. **[1995 - 5 Marks]**

54. A function $f: R\to R$ satisfies the equation $f(x+y)=f(x)f(y)$ for all x, y in R and $f(x)\ne 0$ for any x in R. Let the function be differentiable at $x=0$ and $f'(0)=2$. Show that $f'(x)=2f(x)$ for all x in R. Hence, determine $f(x)$. **[1990 - 4 Marks]**

55. Draw a graph of the function $y=[x]+|1-x|$, $-1\le x\le 3$.

Determine the points, if any, where this function is not differentiable. **[1989 - 4 Marks]**

56. Let $f(x)$ be a function satisfying the condition $f(-x)=f(x)$ for all real x. If $f'(0)$ exists, find its value. **[1987 - 2 Marks]**

57. Let $f(x)$ be defined in the interval $[-2, 2]$ such that

$f(x)=\begin{cases} -1, -2\le x\le 0 \\ x-1, 0<x\le 2 \end{cases}$

and $g(x)=f(|x|)+|f(x)|$

Test the differentiability of $g(x)$ in $(-2, 2)$. **[1986 - 5 Marks]**

58. Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max\{f(t); 0 \le t \le x\}, & 0 \le x \le 1 \\ 3 - x & 1 \le x \le 2 \end{cases}$$ **[1985 - 5 Marks]**

Discuss the continuity and differentiability of the function g(x) in the interval (0, 2).

59. Let $f(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x < 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & 1 \le x \le 2 \end{cases}$ **[1983 - 2 Marks]**

Discuss the continuity of f, f' and f'' on [0, 2].

60. Find the derivative of

$$f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5} & \text{when } x \ne 1 \\ -\dfrac{1}{3} & \text{when } x = 1 \end{cases}$$

at $x = 1$ **[1979]**

Topic-3: Chain Rule of Differentiation, Differentiation of Explicit & Implicit Functions, Parametric & Composite Functions, Logarithmic & Exponential Functions, Inverse Functions, Differentiation by Trigonometric Substitution

1 MCQs with One Correct Answer

1. If y is a function of x and $\log(x + y) - 2xy = 0$, then the value of $y'(0)$ is equal to **[2004S]**
   (a) 1 (b) –1 (c) 2 (d) 0

2. If $y = (\sin x)^{\tan x}$, then $\dfrac{dy}{dx}$ is equal to **[1994]**
   (a) $(\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$
   (b) $\tan x (\sin x)^{\tan x - 1} \cdot \cos x$
   (c) $(\sin x)^{\tan x} \sec^2 x \log \sin x$
   (d) $\tan x (\sin x)^{\tan x - 1}$

3. There exist a function $f(x)$, satisfying $f(0) = 1, f'(0) = -1$, $f(x) > 0$ for all x, and **[1982 - 2 Marks]**
   (a) $f''(x) > 0$ for all x
   (b) $-1 < f''(x) < 0$ for all x
   (c) $-2 \le f''(x) \le -1$ for all
   (d) $f''(x) < -2$ for all x

2 Integer Value Answer/ Non-Negative Integer

4. Let $f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$.

Then the value of $\dfrac{d}{d(\tan\theta)}(f(\theta))$ is **[2011]**

5. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[2009]**

4 Fill in the Blanks

6. If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\dfrac{dy}{dx} =$ .... **[1996 - 1 Mark]**

7. If $f(x) = |x - 2|$ and $g(x) = f[f(x)]$, then $g'(x) =$ .... for $x > 20$ **[1990 - 2 Marks]**

8. The derivative of $\sec^{-1}\left(\dfrac{1}{2x^2 - 1}\right)$ with respect to $\sqrt{1 - x^2}$ at $x = \dfrac{1}{2}$ is .... **[1986 - 2 Marks]**

9. If $f(x) = \log_x(\ln x)$, then $f'(x)$ at $x = e$ is .... **[1985 - 2 Marks]**

10. If $f_r(x), g_r(x), h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$

and $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$ then $F'(x)$ at $x = a$ is ....

[1985 - 2 Marks]

11. If $y=f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x)=\sin x^2$, then $\frac{dy}{dx}=$ .... [1982 - 2 Marks]

5 True / False

12. The derivative of an even function is always an odd function. [1983 - 1 Mark]

6 MCQs with One or More Than One Correct

13. For any positive integer n, define $f_n:(0,\infty)\to\mathbb{R}$ as

$f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$ for all $x\in(0,\infty)$.

(Here, the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE? [Adv. 2018]

(a) $\sum_{j=1}^{5}\tan^2(f_j(0))=55$

(b) $\sum_{j=1}^{10}(1+f_j'(0))\sec^2(f_j(0))=10$

(c) For any fixed positive integer n, $\lim_{x\to\infty}\tan(f_n(x))=\frac{1}{n}$

(d) For any fixed positive integer n, $\lim_{x\to\infty}\sec^2(f_n(x))=1$

14. For every twice differentiable function $f:\mathbb{R}\to[-2,2]$ with $(f(0))^2+(f'(0))^2=85$, which of the following statement(s) is (are) TRUE? [Adv. 2018]

(a) There exist $r,s\in\mathbb{R}$, where $r<s$, such that f is one-one on the open interval (r,s)

(b) There exists $x_0\in(-4,0)$ such that $|f'(x_0)|\le 1$

(c) $\lim_{x\to\infty}f(x)=1$

(d) There exists $\alpha\in(-4,4)$ such that $f(\alpha)+f''(\alpha)=0$ and $f'(\alpha)\neq 0$

15. Let $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be differentiable functions such that $f(x)=x^3+3x+2$, $g(f(x))=x$ and $h(g(g(x)))=x$ for all $x\in\mathbb{R}$. Then [Adv. 2016]

(a) $g'(2)=\frac{1}{15}$ (b) $h'(1)=666$

(c) $h(0)=16$ (d) $h(g(3))=36$

9 Assertion and Reason / Statement Type Questions

16. Let $f(x)=2+\cos x$ for all real x.

STATEMENT - 1 : For each real t, there exists a point c in $[t,t+\pi]$ such that $f'(c)=0$ because

STATEMENT - 2 : $f(t)=f(t+2\pi)$ for each real t. [2007 -3 marks]

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True.

10 Subjective Problems

17. If $y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{c}{x-c}+1$, prove that $\frac{y'}{y}=\frac{1}{x}\left(\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}\right)$. [1998 - 8 Marks]

18. Find $\frac{dy}{dx}$ at $x=-1$, when

$(\sin y)^{\sin\left(\frac{\pi}{2}x\right)}+\frac{\sqrt{3}}{2}\sec^{-1}(2x)+2^x\tan(\ln(x+2))=0$ [1991 - 4 Marks]

19. If $x=\sec\theta-\cos\theta$ and $y=\sec^n\theta-\cos^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2=n^2(y^2+4)$ [1989 - 2 Marks]

20. If α be a repeated root of a quadratic equation $f(x)=0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix}A(x) & B(x) & C(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$ is divisible by $f(x)$, where prime denotes the derivatives. [1984 - 4 Marks]

21. Let $y=e^{x\sin x^3}+(\tan x)^x$. Find $\frac{dy}{dx}$ [1981 - 2 Marks]

22. Given $y=\frac{5x}{3\sqrt{(1-x)^2}}+\cos^2(2x+1)$; Find $\frac{dy}{dx}$. [1980]

Topic-4: Differentiation of Infinite Series, Successive Differentiation, nth Derivative of Some Standard Functions, Leibnitz's Theorem, Rolle's Theorem, Lagrange's Mean Value Theorem

1 MCQs with One Correct Answer

1. Let $g(x) = \log f(x)$ where $f(x)$ is twice differentible positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$. Then, for $N = 1, 2, 3, \ldots\ldots\ldots$ [2008]

$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N+1)^2}\right\}$

2. $\frac{d^2x}{dy^2}$ equals [2007 -3 marks]

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

3. If $f(x)$ is a twice differentiable function and given that $f(1) = 1; f(2) = 4, f(3) = 9$, then [2005S]

(a) $f''(x) = 2$ for $\forall\ x \in (1, 3)$

(b) $f''(x) = f'(x) = 5$ for some $x \in (2, 3)$

(c) $f''(x) = 3$ for $\forall\ x \in (2, 3)$

(d) $f''(x) = 2$ for some $x \in (1, 3)$

4. If $x^2 + y^2 = 1$ then [2000]

(a) $yy'' - 2(y')^2 + 1 = 0$ (b) $yy'' + (y')^2 + 1 = 0$

(c) $yy'' + (y')^2 - 1 = 0$ (d) $yy'' + 2(y')^2 + 1 = 0$

5. Let $f(x)$ be a quadratic expression which is positive for all the real values of x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x, [1990 - 2 Marks]

(a) $g(x) < 0$ (b) $g(x) > 0$

(c) $g(x) = 0$ (d) $g(x) \geq 0$

6. If $y^2 = P(x)$, a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ equals [1988 - 2 Marks]

(a) $P'''(x) + P'(x)$ (b) $P''(x)P'''(x)$

(c) $P(x)P'''(x)$ (d) a constant

2 Integer Value Answer/ Non-Negative Integer

7. For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficients defined by

$$S = \{(x^2-1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____ [Adv 2019]

6 MCQs with One or More Than One Correct

8. Let S be the set of all twice differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $\frac{d^2f}{dx^2}(x) > 0$ for all $x \in (-1, 1)$.

For $f \in S$, let X_f be the number of points $x \in (-1,1)$ for which $f(x) = x$. Then which of the following statements is(are) true? [Adv. 2023]

(a) There exists a function $f \in S$ such that $f X_f = 0$

(b) For every function $f \in S$, we have $X_f \leq 2$

(c) There exists a function $f \in S$ such that $X_f = 2$

(d) There does NOT exist any function f in S such that $X_f = 1$

9. Let $f : (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that

$\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$ for all $x \in (0, \pi)$.

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE? [Adv. 2018]

(a) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(b) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(c) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(d) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

9 Assertion and Reason / Statement Type Questions

10. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$. $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x) \sin x$

STATEMENT - 1 : $\lim_{x \to 0} [g(x) \cot x - g(0) \operatorname{cosec} x] = f''(0)$ and

STATEMENT - 2 : $f'(0) = g(0)$ [2008]

(a) Statement - 1 is True, Statement - 2 is True; Statement - 2 is a correct explanation for Statement - 1

(b) Statement - 1 is True, Statement - 2 is True; Statement - 2 is NOT a correct explaination for Statement - 1

(c) Statement - 1 is True, Statement - 2 is False

(d) Statement - 1 is False, Statement - 2 is True

10 Subjective Problems

11. Let f be a twice differentiable function such that

$f''(x) = -f(x)$, and $f'(x) = g(x)$,

$h(x) = [f(x)]^2 + [g(x)]^2$

Find $h(10)$ if $h(5) = 11$ [1982 - 3 Marks]

Answer Key

Topic-1 : Continuity

1. (d) **2.** (c) **3.** I – {–1,0} **4.** (7) **5.** (a,c,d) **6.** (a,d) **7.** (b,d) **8.** (b,c) **9.** (b)

Topic-2 : Differentiability

1. (b) **2.** (b) **3.** (c) **4.** (a) **5.** (b) **6.** (a) **7.** (d) **8.** (d) **9.** (d) **10.** (a)
11. (d) **12.** (b) **13.** (a) **14.** (c) **15.** (d) **16.** (4) **17.** (3) **18.** (6) **19.** R – {0}
20. (0) **21.** (a,c,d) **22.** (a, b) **23.** (a,c)
24. (a,c,d) **25.** (b,c) **26.** (b,c) **27.** (a,b) **28.** (a,d) **29.** (a,b,c,d) **30.** (b,c) **31.** (a,d) **32.** (a,c,d)
33. (a,c) **34.** (a,b) **35.** (b,c,d) **36.** (a,b,c) **37.** (a) **38.** (a,b,d) **39.** (b,d,e) **40.** (b) **41.** (a,b,d)
42. (c) **43.** (d) **44.** (d) **45.** (A)-p, q, r; (B)-p, s; (C)-s, r; (D)-p, q **46.** (A)-p; (B)-r

Topic-3 : Chain Rule of Differentiation, Differentiation of Explicit & Implicit Functions, Parametric & Composite Functions, Logarithmic & Exponential Functions, Inverse Functions, Differentiation by Trigonometric Substitution

1. (a) **2.** (a) **3.** (a) **4.** (1) **5.** (2) **6.** $\frac{dy}{dx} = 1$ **7.** $g'(x) = -4$

8. $\left.\frac{du}{dv}\right|_{x=\frac{1}{2}} = 4$ **9.** $\frac{1}{e}$ **10.** (0) **11.** $\frac{2+2x-2x^2}{(x^2+1)^2} \sin\left(\frac{2x-1}{x^2+1}\right)^2$ **12.** True **13.** (d) **14.** (a,b,d)
15. (b,c) **16.** (b)

Topic-4 : Differentiation of Infinite Series, Successive Differentiation, nth Derivative of Some Standard Functions, Leibnitz's Theorem, Rolle's Theorem, Lagrange's Mean Value Theorem

1. (a) **2.** (d) **3.** (d) **4.** (b) **5.** (b) **6.** (c) **7.** (5.00) **8.** (a, b, c) **9.** (b,c,d) **10.** (a)

Hints & Solutions

Topic-1: Continuity

1. **(d)** We have $f(x) = [x]^2 - [x^2]$

At $x = 0$,

L.H.L. = $\lim_{h\to 0} f(-h) = \lim_{h\to 0}\left([-h]^2 - [(-h)^2]\right)$

$= \lim_{h\to 0}\left((-1)^2 - [h^2]\right) = \lim_{h\to 0}(1-0) = 1$

R.H.L. = $\lim_{h\to 0} f(h) = \lim_{h\to 0}\left([h]^2 - [h^2]\right) = \lim_{h\to 0}(0-0) = 0$

$\because$ L.H.L. $\neq$ R.H.L.

$\therefore$ $f(x)$ is not continuous at $x = 0$.

At $x = 1$

L.H.L. = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left([1-h]^2 - [(1-h)^2]\right)$

$= \lim_{h\to 0}(0-0) = 0$

R.H.L. = $\lim_{h\to 0} f(1+h) = \lim_{h\to 0}\left([1+h]^2 - [(1+h)^2]\right)$

$= \lim_{h\to 0}(1-1) = 0$

$f(1) = [1]^2 - [1^2] = 1 - 1 = 0$

$\therefore$ L.H.L. = R.H.L. = $f(1)$

$\therefore$ $f(x)$ is continuous at $x = 1$.

Clearly $f(x)$ is not continuous at other integral points.

2. **(c)** $f(x) = [x]\cos\left(\frac{2x-1}{2}\right)\pi$

When x is not an integer, both the functions $[x]$ and $\cos\left(\frac{2x-1}{2}\right)\pi$ are continuous.

$\therefore$ $f(x)$ is continuous on all non integral points.

For $x = n \in I$

LHL = $\lim_{x\to n^-} f(x) = \lim_{x\to n^-}[x]\cos\left(\frac{2x-1}{2}\right)\pi$

$= (n-1)\cos\left(\frac{2n-1}{2}\right)\pi = 0$

RHL = $\lim_{x\to n^+} f(x) = \lim_{x\to n^+}[x]\cos\left(\frac{2x-1}{2}\right)\pi$

$= n\cos\left(\frac{2n-1}{2}\right)\pi = 0$

Also $f(n) = n\cos\frac{(2n-1)\pi}{2} = 0$

Thus LHL = RHL = $f(x)$

$\therefore$ f is continuous at all integral point.

Hence, f is continuous everywhere.

3. Clearly the given function is not defined for those values of x for which $[x+1] = 0$.

i.e., $0 \le x+1 < 1 \Rightarrow -1 \le x < 0$

$\therefore$ Required domain is R – [–1, 0)

We know that $[x]$ is discontinuous at all integral value of x and $\sin\left(\frac{\pi}{[x+1]}\right)$ is discontinuous for $[x+1] = 0$

$\Rightarrow 0 \le x+1 < 1 \Rightarrow -1 \le x < 0$

i.e., [–1, 0)

Also domain of $f = R - [1, 0)$

Hence points of discontinuity of f in their domain

$= I - \{-1, 0\}$

4. $f(x)$ will be continuous at $x = 2$, if

$\lim_{x\to 2} f(x) = f(2) \Rightarrow \lim_{x\to 2}\frac{x^3 + x^2 - 16x + 20}{(x-2)^2} = k$

$\Rightarrow k = \lim_{x\to 2}\frac{(x-2)^2(x+5)}{(x-2)^2} = \lim_{x\to 2}(x+5) = 7$

5. **(a, c, d)** Given : $f(x) = x\cos(\pi(x + [x]))$

Let x = n be any integer not equal to zero.

Then $\lim_{x\to n^-} x\cos(\pi(x+[x])) = n\cos(\pi(n+n-1))$

$= n\cos(2n-1)\pi = -n$

and $\lim_{x\to n^+} x\cos(\pi(x+[x])) = n\cos(\pi(n+[n])) = n\cos 2n\pi = n$

$\therefore$ LHL $\neq$ RHL

$\Rightarrow$ f is discontinuous at x = –1, 1, 2

At x = 0, LHL = RHL = 0 = f(0)

$\therefore$ f is continuous at x = 0.

6. **(a, d)** Let f and g be maximum at c_1 and c_2 respectively, $c_1, c_2 \in [0,1]$

Then, $f(c_1) = g(c_2)$

Let $h(x) = f(x) - g(x)$

Then, $h(c_1) = f(c_1) - g(c_1) > 0$

and $h(c_2) = f(c_2) - g(c_2) < 0$

$\therefore$ $h(x) = 0$ has atleast one root in $[c_1, c_2]$

i.e. $f(c) = g(c)$ for $c \in [c_1, c_2]$,

which shows that options (a) and (d) are correct.

7. **(b, d)** Given : $f(x) = \begin{cases} a_n + \sin\pi x, & x\in[2n, 2n+1] \\ b_n + \cos\pi x, & x\in(2n-1, 2n)\end{cases}$

$\because$ f is continuous for all n

$\therefore$ At $x = 2n$, LHL = RHL = $f(2n)$

$\Rightarrow b_n + \cos 2\pi n = a_n + \sin 2\pi n = a_n + \sin 2\pi n$

$\Rightarrow b_n + 1 = a_n \Rightarrow a_n - b_n = 1$, $\therefore$ option (b) is correct.

Also at $x = 2n+1$, LHL = RHL = $f(2n+1)$

$\Rightarrow \lim_{h\to 0} a_n + \sin\pi(2n+1-h)$

$= \lim_{h\to 0} b_{n+1} + \cos\pi(2n+1-h) = a_n + \sin(2n+1)\pi$

$\Rightarrow a_n = b_{n+1} - 1 = a_n \Rightarrow a_n - b_{n+1} = -1$

$\therefore$ option (c) is incorrect.

$\Rightarrow a_{n-1} - b_n = -1$, $\therefore$ option (d) is correct.

8. **(b, c)** On $(0, \pi)$

(a) $f(x) = \tan x$

We know that $\tan x$ is discontinuous at $x = \pi/2$

(b) $f(x) = \int_0^x t\sin\left(\frac{1}{t}\right)dt$

$\Rightarrow \quad f'(x) = x\sin\left(\frac{1}{x}\right)$, which exists on $(0, \pi)$

$\because \quad f(x)$ is differentiable, on $(0, \pi)$, therefore it is continuous on $(0, \pi)$.

(c) $f(x) = \begin{cases} 1 & , 0 < x \le 3\pi/4 \\ 2\sin\frac{2x}{9} & , 3\pi/4 < x < \pi \end{cases}$

Clearly $f(x)$ may or may not be continuous at $x = \frac{3\pi}{4}$ but it is continuous on $(0, \pi)$ except at $x = \frac{3\pi}{4}$.

L.H.L. $= \lim_{h\to 0} f\left(\frac{3\pi}{4} - h\right) = \lim_{x\to 0} 1 = 1$

R.H.L. $= \lim_{h\to 0} f\left(\frac{3\pi}{4} + h\right) = \lim_{x\to 0} 2\sin\frac{2}{9}\left(\frac{3\pi}{4} + h\right)$

$= \lim_{h\to 0} 2\sin\left(\frac{\pi}{6} + \frac{2h}{9}\right) = 2\sin\frac{\pi}{6} = 1$

Also $f\left(\frac{3\pi}{4}\right) = 1$

$\because$ L.H.L. = R.H.L. $= f\left(\frac{3\pi}{4}\right)$

$\therefore f(x)$ is continuous at $x = \frac{3\pi}{4}$ and hence it is continuous on $(0, \pi)$

(d) $f(x) = \begin{cases} x\sin x & , 0 < x \le \pi/2 \\ \frac{\pi}{2}\sin(\pi + x) & , \frac{\pi}{2} < x < \pi \end{cases}$

Clearly $f(x)$ may or may not be continuous at $x = \frac{\pi}{2}$ but it is continuous on $(0, \pi)$ except at $x = \pi/2$.

At $x = \pi/2$, L.H.L. $= \lim_{h\to 0} f\left(\frac{\pi}{2} - h\right)$

$= \lim_{h\to 0}\left(\frac{\pi}{2} - h\right)\sin\left(\frac{\pi}{2} - h\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$

R.H.L. $= \lim_{h\to 0} f\left(\frac{\pi}{2} + h\right) = \lim_{h\to 0}\frac{\pi}{2}\sin\left(\pi + \frac{\pi}{2} + h\right)$

$= \frac{\pi}{2}\sin\left(\pi + \frac{\pi}{2}\right) = \frac{-\pi}{2}\sin\frac{\pi}{2} = -\frac{\pi}{2}$

Thus, L.H.L. $\ne$ R.H.L., $\therefore f(x)$ is not continuous on $(0, \pi)$.

9. (b) $f(x) = \frac{x}{2} - 1$

$\therefore \quad [f(x)] = \left[\frac{x}{2} - 1\right] = -1$, if $0 \le x < 2$

$\tan[f(x)] = \begin{cases} \tan(-1), & 0 \le x < 2 \\ 0, & 2 \le x \le \pi \end{cases}$

$\therefore$ The function $\tan[f(x)]$ is discontinuous at $x = 2$.

Also the function $\frac{1}{f(x)} = \frac{1}{\frac{x}{2} - 1} = \frac{2}{x-2}$ is discontinuous at $x = 2$.

Thus both the given functions $\tan[f(x)]$ as well as $\frac{1}{f(x)}$ are discontinuous on the interval $[0, \pi]$.

Now, $f^{-1}(x) = y \Rightarrow x = f(y) = \frac{y}{2} - 1 \Rightarrow y = 2(x+1)$

$\therefore \quad f^{-1}(x) = 2(x+1)$ is continuous on $[0, \pi]$

10. Given : $f(x) = \begin{cases} (1 + |\sin x|)^{\frac{a}{|\sin x|}}, & \frac{-\pi}{6} < x < 0 \\ b & , x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & , 0 < x < \frac{\pi}{6} \end{cases}$

is continuous at $x = 0$

$\therefore \quad \lim_{h\to 0} f(0-h) = f(0) = \lim_{h\to 0} f(0+h)$

Now, $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}[1 + |\sin(-h)|]^{\frac{a}{|\sin(-h)|}}$

$= \lim_{h\to 0}[1 + \sin h]^{\frac{a}{\sin h}} \Rightarrow e^{\lim_{h\to 0}\frac{a}{\sin h}\log(1+\sin h)} = e^a$

and $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} e^{\frac{\tan 2h}{\tan 3h}}$

$= e^{\lim_{h\to 0}\frac{\tan 2h}{2h}\times\frac{3h}{\tan 3h}\times\frac{2}{3}} = e^{\frac{2}{3}}$

Also $f(0) = b$

$\therefore \quad e^a = b = e^{\frac{2}{3}} \Rightarrow a = \frac{2}{3}$ and $b = e^{\frac{2}{3}}$

11. Given : $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & , x < 0 \\ a & , x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & , x > 0 \end{cases}$

Since $f(x)$ is continuous at $x = 0$,

$\therefore$ L.H.L at $(x = 0) = f(0)$

$= \lim_{h\to 0}\frac{1 - \cos 4(0-h)}{(0-h)^2} = a \Rightarrow \lim_{h\to 0}\frac{1 - \cos 4h}{h^2} = a$

$= \lim_{h\to 0}\frac{2\sin^2 2h}{4h^2}.4 = a \Rightarrow 8 = a$

12. Given :

$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b & , \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & , \frac{\pi}{2} < x \le \pi \end{cases}$

is continuous for $0 \le x \le \pi$.

$\therefore f(x)$ must be continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$

$$\Rightarrow \lim_{x\to\left(\frac{\pi}{4}\right)^-} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \lim_{h\to0} f\left(\frac{\pi}{4}-h\right) = \frac{2\pi}{4}\cot\frac{\pi}{4}+b$$

$$\Rightarrow \lim_{h\to0}\left(\frac{\pi}{4}-h\right)+a\sqrt{2}\lim_{h\to0}\sin\left(\frac{\pi}{4}-h\right)=\frac{\pi}{2}+b$$

$$\Rightarrow \frac{\pi}{4}+a=\frac{\pi}{2}+b \Rightarrow a-b=\frac{\pi}{4} \quad ...(i)$$

Also, $\lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{h\to0} f\left(\frac{\pi}{2}+h\right) = 2.\frac{\pi}{2}\cot\frac{\pi}{2}+b$$

$$\Rightarrow \lim_{h\to0} a\cos 2\left(\frac{\pi}{2}+h\right)-b\sin\left(\frac{\pi}{2}+h\right)=b$$

$$\Rightarrow a\cos\pi - b\sin\frac{\pi}{2}=b \Rightarrow -a-b=b$$

$$\Rightarrow a+2b=0 \quad ...(ii)$$

On solving (i) and (ii), we get $a=\frac{\pi}{6}$ and $b=\frac{-\pi}{12}$.

13. Let $h(x)=f(x)+g(x)$ be continuous.
$\Rightarrow g(x)=h(x)-f(x)$
Now, $h(x)$ and $f(x)$ both are continuous functions.
$\therefore$ $h(x)-f(x)$ must also be continuous. But it contradicts the given statement that $g(x)$ is discontinuous. Therefore our assumption that $f(x)+g(x)$ a continous function is wrong and hence $f(x)+g(x)$ is discontinuous.

14. $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$

$f(f(x))=\begin{cases}1+f(x) & 0\le f(x)\le 2\\ 3-f(x) & 2<f(x)\le 3\end{cases}$

Now $0\le x<2\Rightarrow 1\le x+1\le 3$
$\Rightarrow 1\le f(x)\le 3$
$2<x\le 3\Rightarrow -3\le -x<-2$
$\Rightarrow 0\le 3-x<1\Rightarrow 0\le f(x)<1$
$\Rightarrow 0\le x\le 1\Rightarrow 1\le f(x)\le 2$
$1<x\le 2\Rightarrow 2<f(x)\le 3$
$2<x\le 3\Rightarrow 0\le f(x)<1$

$\Rightarrow f(f(x))=\begin{cases}2+x & 0\le x\le 1\\ 2-x & 1<x\le 2\\ 4-x & 2<x\le 3\end{cases}$

At $x=1$, R.H.L. $=\lim_{h\to0} g(1+h)=\lim_{h\to0} 2-(1+h)=1$
$g(1)=3$, $\therefore$ discontinuous at $x=1$
At $x=2$, R.H.L. $=\lim_{h\to0} g(2+h)=\lim_{h\to0} 4-(2+h)=2$
$g(2)=0$, $\therefore$ discontinuous at $x=2$

15. Given : $f(x+y)=f(x)+f(y)+x, y$
As $f(x)$ is continuous at $x=0$, we have
LHL = RHL = f (0)

$$\Rightarrow \lim_{h\to0} f(0-h)=\lim_{h\to0} f(0+h)=f(0)$$

$$\Rightarrow f(0)+\lim_{h\to0} f(-h)=f(0)+\lim_{h\to0} f(h)=f(0)$$

$$\Rightarrow \lim_{h\to0} f(-h)=\lim_{h\to0} f(h)=0 \quad ...(i)$$

Now let x = a be any arbitrary point then at x = a,

LHL $=\lim_{h\to0} f(a-h)=\lim_{h\to0}[f(a)+f(-h)]$

$=f(a)+\lim_{h\to0} f(-h)=f(a)$ [using (i)]

Similarly, R.H.L. $=\lim_{h\to a} f(a+h)=f(a)$

$\therefore$ $\lim_{h\to0} f(a-h)=\lim_{h\to0} f(a+h)=f(a)$

Hence, f is continuous at x = a. Since a is any arbitrary point,
$\therefore$ f is continuous $\forall x\in R$.

Topic-2: Differentiability

1. **(b)** Given that, $\lim_{t\to x}\frac{t^{10}f(x)-x^{10}f(t)}{t^9-x^9}=1$

By L-Hospital Rule

$$\lim_{t\to x}\frac{10t^9f(x)-f'(t)x^{10}}{9t^8}=1$$

$$\Rightarrow 10x^9f(x)-f'(x)x^{10}=9x^8$$

$$\Rightarrow f'(x)-\frac{10}{x}f(x)=-\frac{9}{x^2}$$

$$\text{IF}=e^{-\int\frac{10}{x}dx}=\frac{1}{x^{10}}$$

$\therefore$ Solution is

$$\frac{y}{x^{10}}=\int-\frac{9}{x^{10}}\times\frac{1}{x^2}dx$$

$$=-9\int x^{-12}dx$$

$$\frac{y}{x^{10}}=\frac{9}{11}x^{-11}+C$$

put $x=1$ and $y=2$, we get $C=\frac{13}{11}$

$$\Rightarrow y=\frac{9}{11x}+\frac{13}{11}x^{10}$$

2. **(b)** $f'(0^+)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to0}\frac{h^2\left|\cos\frac{\pi}{h}\right|}{h}=\lim_{h\to0}h\left|\cos\frac{\pi}{h}\right|$$

$=0\times$ some finite value $=0$

and $f'(0^-) = \lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0}\dfrac{h^2\left|\cos\frac{\pi}{-h}\right|}{-h}$

$= \lim\limits_{h\to 0} -h\left|\cos\dfrac{\pi}{h}\right| = 0\times$ some finite value $= 0$

$\because\ f'(0^+) = f'(0^-)$ $\therefore$ f is differentiable at $x = 0$

Now $f'(2^+) = \lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2\left|\cos\frac{\pi}{2+h}\right| - 4\left|\cos\frac{\pi}{2}\right|}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2\left(\cos\frac{\pi}{2+h}\right)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2}{h}\sin\left(\dfrac{\pi}{2}-\dfrac{\pi}{2+h}\right)$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2}{h}\sin\left(\dfrac{\pi h}{2(2+h)}\right)$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2}{h}\times\dfrac{\sin\left(\frac{\pi h}{2(2+h)}\right)}{\left(\frac{\pi h}{2(2+h)}\right)}\times\dfrac{\pi h}{2(2+h)} = \pi$

and $f'(2^-) = \lim\limits_{h\to 0}\dfrac{f(2-h)-f(2)}{-h}$

$= \lim\limits_{h\to 0}\dfrac{(2-h)^2\left|\cos\left(\frac{\pi}{2-h}\right)\right|-0}{-h}$

$= \lim\limits_{h\to 0}\dfrac{-(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{-h}$

$= \lim\limits_{h\to 0}\dfrac{(2-h)^2\sin\left(\frac{\pi}{2}-\frac{\pi}{2-h}\right)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2-h)^2}{h}\times\dfrac{\sin\left(\frac{-\pi h}{2(2-h)}\right)}{\left(\frac{-\pi h}{2(2-h)}\right)}\times\left(\dfrac{-\pi h}{2(2-h)}\right) = -\pi$

$\because\ f'(2^+)\neq f'(2^-)$, $\therefore$ f is not differentiable at $x = 2$.

3. **(c)** $\because$ p = left hand derivative of $|x-1|$ at $x = 1 \Rightarrow p = -1$

Now $\lim\limits_{x\to 1^+} g(x) = p$, where

$g(x) = \dfrac{(x-1)^n}{\log\cos^m(x-1)}$, $0<x<2$,

m, n are integers, $m\neq 0$, $n>0$

$\therefore\ \lim\limits_{x\to 1^+}\dfrac{(x-1)^n}{\log\cos^m(x-1)} = -1$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{h^n}{\log\cos^m h} = -1$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{h^n}{m(\log\cos h)} = -1$ (using LH rule)

$\Rightarrow \lim\limits_{h\to 0}\dfrac{n h^{n-1}\cos h}{m(-\sin h)} = -1$ (using LH rule)

$\Rightarrow \lim\limits_{h\to 0}\dfrac{n h^{n-2}\cos h}{m\left(\frac{\sin h}{h}\right)} = 1 \Rightarrow n = 2$ and $m = 2$

4. **(a)** Given : $f(x)$ is differentiable on $(0,\infty)$ such that

$f(1) = 1$ and $\lim\limits_{t\to x}\dfrac{t^2 f(x)-x^2 f(t)}{t-x} = 1$ for each $x>0$

$\Rightarrow \lim\limits_{t\to x}\dfrac{2t\,f(x)-x^2 f'(t)}{1} = 1$ (using L' H rule)

$\Rightarrow 2xf(x)-x^2 f'(x) = 1 \Rightarrow f'(x)-\dfrac{2}{x}f(x) = -\dfrac{1}{x^2}$

(Linear differential equation) Integrating factor,

$e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log 1/x^2} = \dfrac{1}{x^2}$

$\therefore$ Solution : $f(x)\times\dfrac{1}{x^2} = \displaystyle\int\left(-\dfrac{1}{x^2}\right)\times\dfrac{1}{x^2}dx$

$\Rightarrow \dfrac{f(x)}{x^2} = \dfrac{1}{3x^3}+c \Rightarrow f(x) = cx^2+\dfrac{1}{3x}$

$\because\ f(1) = 1$,

$\therefore\ 1 = c+\dfrac{1}{3} \Rightarrow c = 2/3$

$\therefore\ f(x) = \dfrac{2}{3}x^2+\dfrac{1}{3x}$

5. **(b)** Given : $f(x)$ is a continuous and differentiable function and

$f\left(\dfrac{1}{n}\right) = 0$, $\forall n\geq 1$ and $n\in I$

$\therefore\ f(0^+) = f\left(\dfrac{1}{\infty}\right) = 0$

$\because$ R.H.L. $= 0$,

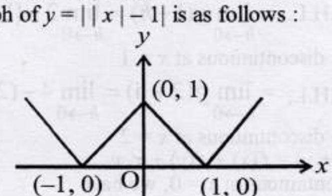
$\therefore$ $f(0) = 0$ for $f(x)$ to be continuous.

Also $f'(0) = \lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h-0} = \lim\limits_{h\to 0}\dfrac{f(h)}{h} = 0$

$= 0$ (using $f(0) = 0$ and $f(0^+) = 0$)

$\therefore$ $f(0) = 0, f'(0) = 0$

6. **(a)** Graph of $y = ||x|-1|$ is as follows :



The graph has sharp turnings at $x = -1, 0$. Therefore given function is not differentiable at $x = -1, 0, 1$.

7. (d) $f(x)=\begin{cases}\tan^{-1}x, & \text{if } |x|\le 1\\ \frac{1}{2}(|x|-1), & \text{if } |x|>1\end{cases}$

$\Rightarrow f(x)=\begin{cases}\frac{1}{2}(-x-1), & \text{if } x<-1\\ \tan^{-1}x, & \text{if } -1\le x\le 1\\ \frac{1}{2}(x-1), & \text{if } x>1\end{cases}$

Clearly L.H.L. at $(x=-1)=\lim_{h\to 0} f(-1-h)=0$

R.H.L. at $(x=-1)=\lim_{h\to 0} f(-1+h)$

$=\lim_{h\to 0}\tan^{-1}(-1+h)=3\pi/4$

∴ At $x=-1$, L.H.L. ≠ R.H.L.

∴ $f(x)$ is discontinuous at $x=-1$

Also we can prove in the same way, that $f(x)$ is discontinuous at $x=1$

∴ $f'(x)$ can not be found for $x=\pm 1$

Hence, domain of $f'(x)=R-\{-1,1\}$

8. (d) Let us test each of four options :

(a) $f(x)=\cos|x|+|x|=\begin{cases}\cos x-x, & x<0\\ \cos x+x, & x\ge 0\end{cases}$

$f'(x)=\begin{cases}-\sin x-1, & x<0\\ -\sin x+1, & x\ge 0\end{cases}$

At $x=0$, LHD $=-1$, RHD $=1$

∴ $f(x)$ is not differentiable.

(b) $f(x)=\cos|x|-|x|=\begin{cases}\cos x+x, & x<0\\ \cos x-x, & x\ge 0\end{cases}$

∴ $f(x)$ is not differentiable at $x=0$

(c) $f(x)=\sin|x|+|x|=\begin{cases}-\sin x-x, & x<0\\ \sin x+x, & x\ge 0\end{cases}$

∴ $f(x)$ is not differentiable at $x=0$

(d) $f(x)=\sin|x|-|x|=\begin{cases}-\sin x+x, & x<0\\ \sin x-x, & x\ge 0\end{cases}$

$f'(x)=\begin{cases}-\cos x+1, & x<0\\ \cos x-1, & x\ge 0\end{cases}$

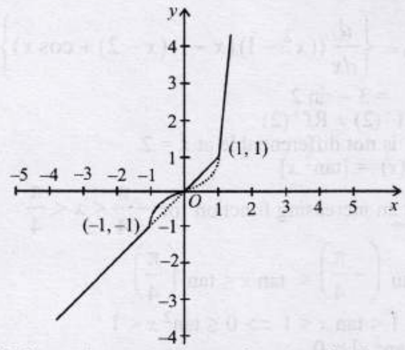
At $x=0$, LHD $=0$, RHD $=0$

∴ $f(x)$ is differentiable at $x=0$.

9. (d) $f(x)=\max.\{x,x^3\}$

$=\begin{cases}x; & x<-1\\ x^3; & -1\le x\le 0\\ x; & 0\le x\le 1\\ x^3; & x\ge 1\end{cases}$

Graph of $f(x)=\max\{x,x^3\}$ is as shown with solid lines.



We know that a continuous function $f(x)$ is not differentiable at $x=a$ if graphically it takes a sharp turn at $x=a$. Since, in the graph there are sharp turns at $x=-1, 0, 1$;

∴ $f(x)$ is not differentiable at $x=-1, 0, 1$.

10. (a) $\text{LHD}_{\text{at }x=k}=\lim_{h\to 0}\dfrac{f(k)-f(k-h)}{h}$ (k = integer)

$=\lim_{h\to 0}\dfrac{[k]\sin k\pi-[k-h]\sin(k-h)\pi}{h}$

$=\lim_{h\to 0}\dfrac{-(k-1)\sin(k-h)\pi}{h}$ $[\because \sin k\pi=0]$

$=\lim_{h\to 0}\dfrac{-(k-1)\sin(k\pi-h\pi)}{h}$

$=\lim_{h\to 0}\dfrac{-(k-1)(-1)^{k-1}\sin h\pi}{h\pi}\times\pi$

$[\because \sin(k\pi-\theta)=(-1)^{k-1}\sin\theta]$

$=\pi(k-1)(-1)^k$

11. (d) Since $|x|=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\ge 0\end{cases}$

$|x^2-3x+2|=|(x-1)(x-2)|$

$=\begin{cases}(x-1)(x-2), & \text{if } x<1\\ (x-1)(2-x), & \text{if } 1\le x<2\\ (x-1)(x-2), & \text{if } x\ge 2\end{cases}$

and $\cos(-\theta)=\cos\theta \Rightarrow \cos|x|=\cos x$

∴ $f(x)=\begin{cases}(x^2-1)(x-1)(x-2)+\cos x, & \text{if } x\le 1\\ -(x^2-1)(x-1)(x-2)+\cos x, & \text{if } 1\le x<2\\ (x^2-1)(x-1)(x-2)+\cos x, & \text{if } x\ge 2\end{cases}$

This function may or may not be differentiable at $x=1$ and $x=2$ but is differentiable at all points except at $x=1$ and $x=2$. Let us check the differentiability at $x=1$ and $x=2$.

$Lf'(1)=\left\{\dfrac{d}{dx}[(x^2-1)(x-1)(x-2)+\cos x]\right\}_{x=1}=-\sin 1$

$Rf'(1)=\left\{\dfrac{d}{dx}[-(x^2-1)(x-1)(x-2)+\cos x]\right\}_{x=1}=-\sin 1$

∵ $Lf'(1)=Rf'(1)$

∴ f is differentiable at x = 1.

$Lf'(2)=\left\{\dfrac{d}{dx}[-(x^2-1)(x-1)(x-2)+\cos x]\right\}_{x=2}$

$=-3-\sin 2$

$$Rf'(2) = \left\{\frac{d}{dx}((x^2-1)(x-1)(x-2)+\cos x)\right\}_{x=2}$$
$= 3 - \sin 2$

$\because$ $Lf'(2) \neq Rf'(2)$

$\therefore$ f is not differentiable at x = 2.

12. **(b)** $f(x) = [\tan^2 x]$

$\tan x$ is an increasing function for $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$\therefore$ $\tan\left(-\frac{\pi}{4}\right) < \tan x < \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow$ $-1 < \tan x < 1 \Rightarrow 0 \le \tan^2 x < 1$

$\Rightarrow$ $[\tan^2 x] = 0$

$\therefore$ $\lim_{x\to 0} f(x) = \lim_{x\to 0}[\tan^2 x] = 0, \; f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$

13. **(a)** $f: R \to R$ is a differentiable function and $f(1) = 4$

$$\lim_{x\to 1}\int_4^{f(x)} \frac{2t}{x-1}dt = \lim_{x\to 1}\left[\frac{t^2}{x-1}\right]_4^{f(x)}$$

$$= \lim_{x\to 1}\frac{(f(x))^2-16}{x-1} = \lim_{x\to 1}\frac{f(x)-4}{x-1}.\lim_{x\to 1}(f(x)+4)$$

$$= \lim_{x\to 1} f'(x).\lim_{x\to 1}(f(x)+4)$$

$$= f'(1).(f(1)+4) = 8f'(1) \;[\because f(1) = 4]$$

14. **(c)** $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$

$$= \lim_{h\to 0}\frac{g(a+h)f(a)-g(a)f(a+h)}{h}$$

$$= \lim_{h\to 0}\frac{g(a+h)f(a)-g(a)f(a)+g(a)f(a)-g(a)f(a+h)}{h}$$

$$= \lim_{h\to 0} f(a)\left[\frac{g(a+h)-g(a)}{h}\right] - \lim_{h\to 0} g(a)\left[\frac{f(a+h)-f(a)}{h}\right]$$

$= f(a)\, g'(a) - g(a)\, f'(a) = 2 \times 2 - (-1) \times 1 = 5$

15. **(d)** Given : $f(x) = \frac{\tan(\pi[x-\pi])}{1+[x]^2}$

Clearly $[x - \pi]$ is an integer whatever be the value of x.

$\therefore$ $\pi[x - \pi]$ is an integral multiple of π.

Consequently $\tan(\pi[x - \pi]) = 0, \forall$ x.

Also $1 + [x]^2 \neq 0$ for any x.

$\therefore$ $f(x) = 0$.

Hence, $f(x)$ is constant function and therefore, it is continuous and differentiable any number of times, that is $f'(x), f''(x), f'''(x)$,... all exist for every x, their value being 0 at every point x. Hence, out of all the alternatives only (d) is correct.

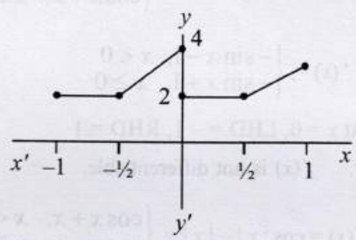
16. **(4)** Given that $f(x) = |2x - 1| + |2x + 1|$

and $g(x) = x - [x] = \{x\}$

$\therefore$ $(fog)(x) = |2\{x\} - 1| + |2\{x\} + 1|$

$$\Rightarrow (fog)(x) = \begin{cases} |2x+1|+|2x+3|, & x\in\left[-1,-\frac{1}{2}\right] \\ |2x+1|+|2x+3|, & x\in\left(-\frac{1}{2},0\right] \\ |2x-1|+|2x+1|, & x\in\left(0,\frac{1}{2}\right] \\ |2x-1|+|2x+1|, & x\in\left(\frac{1}{2},1\right] \end{cases}$$

$$\Rightarrow (fog)(x) = \begin{cases} 2, & x\in\left[-1,-\frac{1}{2}\right) \\ 4x+4, & x\in\left[-\frac{1}{2},0\right] \\ 2, & x\in\left(0,\frac{1}{2}\right] \\ 4x, & x\in\left(\frac{1}{2},1\right) \end{cases}$$



$\therefore$ $f(g(x))$ is discontinuous at $x = 0$.

$\therefore$ $c = 1$

$$\text{Now,}(fog)'(x) = \begin{cases} 0, & x\in\left(-1,\frac{1}{2}\right) \\ 4, & x\in\left(-\frac{1}{2},0\right) \\ 0, & x\in\left(0,\frac{1}{2}\right) \\ 4, & x\in\left(\frac{1}{2},1\right) \end{cases}$$

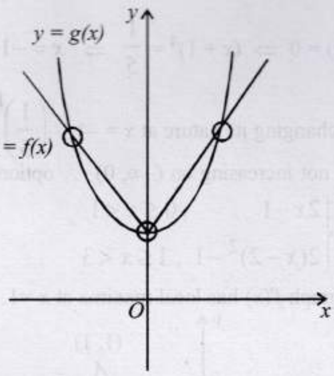
$\therefore$ $f(g(x))$ is non-differentiable at $x = -\frac{1}{2}, 0, \frac{1}{2}$

$\therefore$ d = 3

Hence, c + d = 4

17. **(3)** $f(x) = |x| + 1 = \begin{cases} x+1, & x \ge 0 \\ -x+1, & x < 0 \end{cases}$

$g(x) = x^2 + 1$



From graph, it is clear that there are 3 points at which $h(x)$ is not differentiable.

18. **(6)** $6\int_1^x f(t)dt = 3xf(x) - x^3$

On differentiating, we get $6f(x) = 3f(x) + 3xf'(x) - 3x^2$

$\Rightarrow f'(x) - \frac{1}{x}f(x) = x$, I.F. $= \frac{1}{x}$

$\therefore$ Solution is $f(x).\frac{1}{x} = \int 1.dx = x + c$

$\therefore f(x) = x^2 + cx$

But $f(1) = 2 \Rightarrow c = 1, \quad \therefore f(x) = x^2 + x$

$\Rightarrow f(2) = 4 + 2 = 6$

Note : Putting $x = 1$ in given integral equation, we get

$f(1) = \frac{1}{3}$ while given $f(1) = 2$.

$\therefore$ Data given in the question is inconsistent.

19. We have,

$$f(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases} \Rightarrow f''(x) = \begin{cases} -2, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

Thus $f''(x)$ exists at each point except at $x = 0$

$\therefore f(x)$ is twice differentiable on $R - \{0\}$.

20. Given : $f(x) = \begin{cases} (x-1)^2 \sin\frac{1}{x-1} - |x|, & x \neq 1 \\ -1, & x = 1 \end{cases}$

Since $|x|$ is not differentiable at $x = 0$

$\therefore (x-1)^2 \sin\frac{1}{x-1} - |x|$ is not differentiable at $x = 0$.

At all other values of x, $f(x)$ is differentiable.

$\therefore$ Required set of points is $\{0\}$.

21. **(a, c, d)** Let domain and codomain of function y = f(x) are S and T respectively.

(a) There are infinitely many elements in domain sets S and four elements in codomain set T.

So, there are infinitely many function from S to T.

Hence, option (a) in correct

(b) If number of elements in domain is greater than number of elements in co-domain, then number of strictly increasing function is zero. Hence, option (b) is incorrect.

(c) Since, every subset (0, 1), (1, 2), (3, 4) has four choices.

$\therefore$ Maximum number of continuous functions $= 4 \times 4 \times 4 = 64$ [$\because 64 < 120$]

Hence, option (c) is correct.

(d) Since, every continuous function is piece wise constant functions. f′(x) = 0, so, f(x) is differentiable.

Hence option (d) is correct

22. **(a, b)** $f(x) = \begin{cases} 0 & : 0 < x < \frac{1}{4} \\ \left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right); & \frac{1}{4} \leq x < \frac{1}{2} \\ 2\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right); & \frac{1}{2} \leq x < \frac{3}{4} \\ 3\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right); & \frac{3}{4} \leq x < 1 \end{cases}$

$f\left(\frac{3^-}{4}\right) = \frac{1}{8}$ and $f\left(\frac{3^+}{4}\right) = \frac{3}{16}$

So, f(x) is discontinuous at $x = \frac{3}{4}$ only and

$$f'(x) = \begin{cases} 0 & ; 0 < x < \frac{1}{4} \\ 2\left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right) + \left(x - \frac{1}{4}\right)^2 & ; \frac{1}{4} < x < \frac{1}{2} \\ 4\left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right) + 2\left(x - \frac{1}{4}\right)^2 & ; \frac{1}{2} < x < \frac{3}{4} \\ 6\left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right) + 3\left(x - \frac{1}{4}\right)^2 & ; \frac{3}{4} < x < 1 \end{cases}$$

$\because f'\left(\frac{1^-}{2}\right) \neq f'\left(\frac{1^+}{2}\right)$ and $f'\left(\frac{3^-}{4}\right) \neq f'\left(\frac{3^+}{4}\right)$

f(x) is non-differentiable at $x = \frac{1}{2}$ and $\frac{3}{4}$ and minimum values of f(x) occur at $x = \frac{5}{12}$ whose values is $-\frac{1}{432}$

23. **(a, c)** Let $f : R \to R$ is defined by $f(x) = (x^2 + \sin x)(x - 1)$

Then, $f(1^+) = f(1^-) = f(1) = 0$

Let $(fg) : R \to R$ is defined by $(fg)(x) = f(x).g(x)$

Let $fg(x) = h(x) = f(x).g(x)$ then $h : R \to R$

$h'(x) = f'(x)g(x) + f(x)g'(x)$

If g is differentiable at $x = 1$

$h'(1) = f'(1)g(1) + 0, \qquad [\because f(1) = 0]$

$\Rightarrow$ if $g(x)$ is differentiable then $h(x)$ is also differentiabe (true)

$\Rightarrow$ if $g(x)$ is defferentiable at $x = 1$, then fg is also differentiable at $x = 1$

If $g(x)$ is continous at $x = 1$, then $g(1^+) = g(1^-) = g(1)$

$$h'(1^+) = \lim_{h \to 0^+} \frac{h(1+h) - h(1)}{h}$$

$$h'(1^+) = \lim_{h \to 0^+} \frac{f(1+h)g(1+h) - 0}{h} = f'(1)g(1)$$

$$h'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)g(1-h)-0}{-h} = f'(1)g(1)$$

$\Rightarrow$ $h(x) = f(x).g(x)$ is differentiable at $x = 1$ (True)

So, if g is continous at $x = 1$, then fg is differentiable at $x = 1$.

option (b) (d) $h'(1^+) = \lim\limits_{h\to 0^+} \dfrac{h(1+h)-h(1)}{-h}$

$$h'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$$

$$h'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)g(1-h)}{-h} = f'(1)g(1^-)$$

$\Rightarrow$ $g(1^+) = g(1^-)$

So, it does not mean that if fg is differentiable at $x = 1$, then g is continous or differentiable at $x = 1$

24. (a, c, d)

$$f(x) = \begin{cases} (x^5+5x^4+10x^3+10x^2+5x+1)-2x, & x<0 \\ x^2-2\times\frac{1}{2}\times x+\frac{1}{4}+\frac{3}{4}, & 0\le x<1 \\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3}, & 1\le x<3 \\ (x-2)\log_e(x-2)-x+\frac{10}{3}, & x\ge 3 \end{cases}$$

$$= \begin{cases} (x+1)^5-2x, & x<0 \\ \left(x-\frac{1}{2}\right)^2+\frac{3}{4}, & 0\le x<1 \\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3}, & 1\le x<3 \\ (x-2)\log_e(x-2)-x+\frac{10}{3}, & x\ge 3 \end{cases}$$

For $x = 0$, $f(x) = 1$

For $x < 0$, $f(x) = (x + 1)^5 - 2x$

It decreases to $-\infty$.

$\therefore$ $f(x) \in (-\infty, 1]$ for $x \le 0$

For $x = 3$, $f(x) = \dfrac{1}{3}$

For $x \ge 3$, $f(x)$ increases to ∞

$\therefore$ $f(x) \in \left[\dfrac{1}{3}, \infty\right)$ for $x \ge 3$

On combining the two $f(x) \in R$ $\Rightarrow$ f is onto.

$\therefore$ option (d) is correct.

$$f'(x) = \begin{cases} 5(x+1)^4-2, & x<0 \\ 2x-1, & 0\le x<1 \\ 2(x-2)^2-1, & 1\le x<3 \\ \log_e(x-2), & x\ge 3 \end{cases}$$

$Lf''(1) = 2$, $Rf''(1) = -4$, $\Rightarrow f'$ is not differentiable at x = 1

$\therefore$ option (c) is correct.

For $x < 0$, $f'(x) = 5(x + 1)^4 - 1$

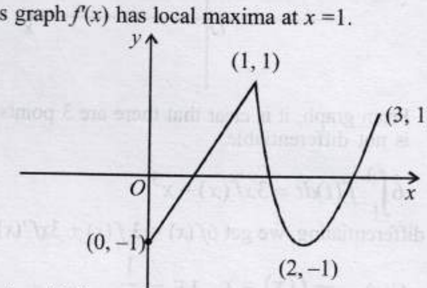
Now, $f'(x) = 0 \Rightarrow (x+1)^4 = \dfrac{1}{5} \Rightarrow x = -1 \pm \left(\dfrac{1}{5}\right)^{1/4}$

$\Rightarrow$ f is changing its nature at $x = -1 - \left(\dfrac{1}{5}\right)^{1/4}$

$\therefore$ f is not increasing on $(-\infty, 0)$ $\therefore$ option (b) is incorrect.

$$f'(x) = \begin{cases} 2x-1, & 0\le x<1 \\ 2(x-2)^2-1, & 1\le x<3 \end{cases}$$

From its graph $f'(x)$ has local maxima at $x = 1$.



$\therefore$ option (a) is correct.

25. (b, c) Given : $f'(x) = e^{(f(x)-g(x))} \cdot g'(x)\ \forall x \in R$

$\Rightarrow$ $e^{-f(x)}f'(x) = e^{-g(x)}g'(x)$

Integrating both sides, we get

$-e^{-f(x)} = -e^{-g(x)} + c \Rightarrow -e^{-f(x)} + e^{-g(x)} = c$

$\Rightarrow$ $-e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)}$

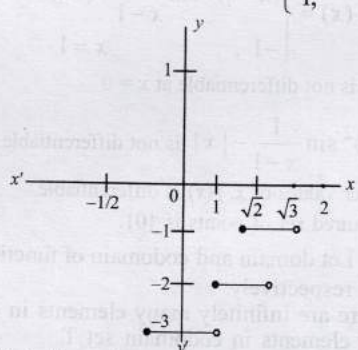
$\therefore -e^{-1} + e^{-g(1)} = -e^{-f(2)} + e^{-1}$ $[\because f(1) = g(2) = 1]$

$\Rightarrow$ $e^{-f(2)} + e^{-g(1)} = \dfrac{2}{e} \Rightarrow e^{-f(2)} < \dfrac{2}{e}$ and $e^{-g(1)} < \dfrac{2}{e}$

$\Rightarrow$ $-f(2) < \ln 2 - 1$ and $-g(1) < \ln 2 - 1$

$\Rightarrow$ $f(2) > 1 - \ln 2$ and $g(1) > 1 - \ln 2$

26. (b, c) $f(x) = [x^2 - 3] = [x^2] - 3 = \begin{cases} -3, & -1/2 \le x < 1 \\ -2, & 1 \le x < \sqrt{2} \\ -1, & \sqrt{2} \le x < \sqrt{3} \\ 0, & \sqrt{3} \le x < 2 \\ 1, & x = 2 \end{cases}$



Clearly, $f(x)$ is discontinuous at 4 points. Option (b) is correct.

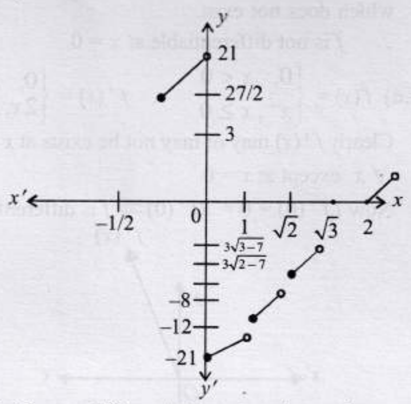
and $g(x) = |x| f(x) + |4x - 7| f(x)$

$= (|x| + |4x - 7|) f(x)$

$= (|x| + |4x - 7|) [x^2 - 3]$

$$= \begin{cases} (-x-4x-7)(-3), & -1/2 \le x < 0 \\ (x-4x+7)(-3), & 0 \le x < 1 \\ (x-4x+7)(-2), & 1 \le x < \sqrt{2} \\ (x-4x+7)(-1), & \sqrt{2} \le x < \sqrt{3} \\ (x-4x+7)(0), & \sqrt{3} \le x < 7/4 \\ (x+4x-7)(0), & 7/4 \le x < 2 \\ (x+4x-7)(1), & x = 2 \end{cases}$$

$$\therefore g(x) = \begin{cases} 15x+21, & -1/2 \le x < 0 \\ 9x-21, & 0 \le x < 1 \\ 6x-14, & 1 \le x < \sqrt{2} \\ 3x-7 & \sqrt{2} \le x < \sqrt{3} \\ 0, & \sqrt{3} \le x < 2 \\ 5x-7, & x = 2 \end{cases}$$



Clearly, g(x) is not differentiable at 4 points, when $x \in (-1/2, 2)$.

$\therefore$ Option (c) is correct.

27. **(a, b)** $f(x) = a\cos(|x^3 - x|) + b\,|x| \sin(|x^3 + x|)$

(a) If a = 0, b = 1
$\Rightarrow f(x) = |x| \sin |x^3 + x|$
$= x \sin(x^3 + x)$
Which is differentiable every where.

(b), (c) If a = 1, b = 0 $\Rightarrow f(x) = \cos(|x^3 - x|) = \cos(x^3 - x)$
Which is differentiable every where.

(d) When a = 1, b = 1, $f(x) = \cos(x^3 - x) + x\sin(x^3 + x)$
Which is differentiable at x = 1
Hence only options (a) and (b) are the correct options.

28. **(a, d)** $f(x) = \begin{cases} \frac{x}{|x|} g(x), x \ne 0 \\ 0 \quad , x = 0 \end{cases} = \begin{cases} -g(x) & , x < 0 \\ 0 & , x = 0 \\ g(x) & , x > 0 \end{cases}$

$$f'(x) = \begin{cases} -g'(x) & , x < 0 \\ 0 & , x = 0 \\ g'(x) & , x > 0 \end{cases}$$

$\therefore$ $Lf'(0) = -g'(0) = 0$ and $Rf'(0) = g'(0) = 0$
$\therefore$ f is differentiable at $x = 0$

$$h(x) = e^{|x|} = \begin{cases} e^{-x} & , x < 0 \\ e^{x} & , x \ge 0 \end{cases};$$

$$h'(x) = \begin{cases} -e^{-x} & , x < 0 \\ e^{x} & , x \ge 0 \end{cases}$$

$\therefore$ $Lh'(0) = -1, Rh'(0) = 1$
$\therefore$ h is not differentiable at $x = 0$
$f\,oh(x) = f(h(x)) = f(e^{|x|})$

$$= \begin{cases} g(e^{-x}) & if \quad x < 0 \\ g(1) & if \quad x = 0 \\ g(e^{x}) & if \quad x > 0 \end{cases}$$

$$f'[h(x)] = \begin{cases} -g'(e^{-x}).e^{-x} & , x < 0 \\ 0 & , x = 0 \\ g'(e^{x})e^{x} & , x > 0 \end{cases}$$

$Lf'(h(0)) = -g'(1), Rf'(h(0)) = g'(1)$
$\because$ $g'(1) \ne 0$, $\therefore Lf'(h(0)) \ne Rf'(h(0))$
$\therefore$ foh is not differentiable at $x = 0$.

$$hof(x) = \begin{cases} e^{|f(x)|} & , x \ne 0 \\ 1 & , x = 0 \end{cases}$$

$$Lh'(f(0)) = \lim_{k \to 0} \frac{h(f(0)) - h(f(0-k))}{k}$$

$$= \lim_{k \to 0} \frac{1 - e^{|g(-k)|}}{k} = \lim_{k \to 0} \frac{1 - e^{|g(-k)|}}{|g(-k)|} \times \frac{|g(-k)|}{k}$$

$$= 1 \times 0 = 0 \left(\because g'(0) = 0 \Rightarrow \lim_{k \to 0} \frac{g(-k)}{k} = \lim_{k \to 0} \frac{g(k)}{k} = 0 \right)$$

$$Rh'(f(0)) = \lim_{k \to 0} \frac{h(f(0+k)) - h(f(0))}{k}$$

$$= \lim_{k \to 0} \frac{e^{|g(k)|} - 1}{k} = \lim_{k \to 0} \frac{e^{|g(k)|} - 1}{|g(k)|} \times \frac{|g(k)|}{k} = 0$$

$\because$ $Lh'(f(0)) = Rh'(f(0)) = 0$
$\therefore$ hof is differentiable at $x = 0$.

29. **(a, b, c, d)** At $x = -\frac{\pi}{2}$, LHL = $\lim_{x \to -\frac{\pi}{2}^-} -x - \frac{\pi}{2} = 0$

and RHL = $\lim_{x \to -\frac{\pi}{2}^+} -\cos x = 0$ and $f\left(\frac{-\pi}{2}\right) = 0$

$\therefore$ LHL = RHL = $f\left(\frac{-\pi}{2}\right)$

$\therefore$ $f(x)$ is continuous at $x = -\frac{\pi}{2}$

At at $x = 0$, $Lf'(0) = \sin 0 = 0$ and $Rf'(0) = 1 - 0 = 1$
$\because$ $Lf'(0) \ne Rf'(0)$
$\therefore$ f is not differentiable at $x = 0$
At $x = 1$, $Lf'(1) = R'f(1)$
$\therefore$ f is differentiable at $x = 1$.

At $x = \frac{-3}{2}$, $f(x) = -\cos x$, which is differentiable.

Hence, all four options are correct.

30. **(b, c)** Given : $f(x+y) = f(x) + f(y)\ \forall\ x, y \in R$
$\therefore$ On putting $x = y = 0$, we get $f(0) = 0$

Also $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

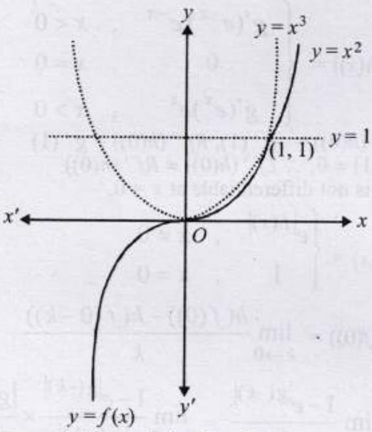
$= \lim_{h \to 0} \frac{f(h)}{h} = f'(0) = k \text{ (say)} \Rightarrow f(x) = kx + c$

But $f(0) = 0 \Rightarrow c = 0, \quad \therefore \; f(x) = kx$

Which is continuous $\forall \; x \in R$.

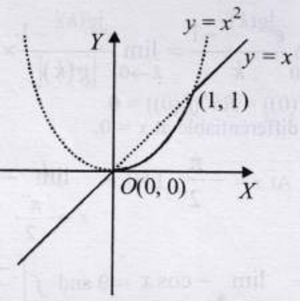
Also $f'(x) = k$, a constant.

31. **(a, d)** From graph, $f(x)$ is continuous everywhere but not differentiable at $x = 1$ as there is sharp turns in the graph at $x = 1$.



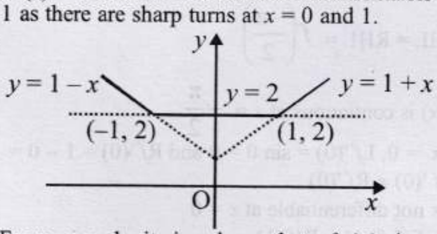
32. **(a, c, d)** From the figure it is clear that

$$h(x) = \begin{cases} x, & \text{if } x \le 0 \\ x^2, & \text{if } 0 < x < 1 \\ x, & \text{if } x \ge 1 \end{cases}$$



From the graph it is clear that h is continuous for all $x \in R$, $h'(x) = 1$ for all $x > 1$ and h is not differentiable at $x = 0$ and 1 as there are sharp turns at $x = 0$ and 1.

33. **(a, c)**



From graph it is clear that $f(x)$ is continuous every where and also differentiable everywhere except at $x = 1$ and -1.

34. **(a, b)** $g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$

If $x \ne 0$, $g'(x) = x^2 \cos(1/x) \left(-\frac{1}{x^2}\right) + 2x \sin\frac{1}{x}$

$= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$, which exists for $\forall x \ne 0$.

If $x = 0$, $g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$

$= \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

$\therefore \quad g'(x) = \begin{cases} -\cos\left(\frac{1}{x}\right) + 2x \sin\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

At $x = 0$, $\cos\left(\frac{1}{x}\right)$ is not continuous, therefore $g'(x)$ is not continuous at $x = 0$.

At $x = 0$, $Lf' = \lim_{x \to 0} \frac{0 - (-x)\sin\left(-\frac{1}{x}\right)}{x} = -\sin\left(\frac{1}{x}\right)$,

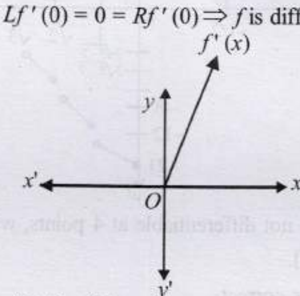
which does not exist.

$\therefore \quad f$ is not differentiable at $x = 0$.

35. **(b,c,d)** $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \ge 0 \end{cases} \quad \therefore \; f'(x) = \begin{cases} 0, & x < 0 \\ 2x, & x \ge 0 \end{cases}$

Clearly $f'(x)$ may or may not be exists at $x = 0$ but it exists $\forall \; x$ except at $x = 0$.

Now $Lf'(0) = 0 = Rf'(0) \Rightarrow f$ is differentiable at $x = 0$



Thus, $f(x)$ is differentiable for all values of x and hence it is continuous also for all values of x.

From graph of $f'(x)$, it is clear that $f'(x)$ is continuous but not differentiable at $x = 0$ as there is sharp turns at $x = 0$ in the graph.

36. **(a,b,c)** $f(x) = \begin{cases} |x - 3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$

$= \begin{cases} \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \\ 3 - x, & 1 \le x < 3 \\ x - 3, & x \ge 3 \end{cases}$

$Lf'(1) = 1$ and $Rf'(1) = -1$.

$\therefore \quad Lf'(1) = Rf'(1)$

Hence, f is differentiable at $x = 1$ and therefore continuous at $x = 1$.

Now, $Lf'(3) = -1$ and $Rf'(3) = 1$

$\therefore \quad Lf'(3) \ne Rf'(3)$

Hence, f is not differentiable at $x = 3$

Now, L.H.L. = $\lim_{h\to 0} f(3-h) = \lim_{h\to 0}[3-(3-h)] = 0$

R.H.L. = $\lim_{h\to 0} f(3+h) = \lim_{h\to 0}[3+h-3] = 0$

and $f(3) = 0$, ∴ LHL = RHL = $f(3)$

Hence, f is continuous at $x = 3$

37. **(a)** $f(x) = \dfrac{x}{1+|x|} = \begin{cases} \dfrac{x}{1-x}, & x<0 \\ \dfrac{x}{1+x}, & x \ge 0 \end{cases}$

Clearly $f(x)$ may or may not be differentiable at $x = 0$ but $f(x)$ differentiable at each pair in $(-\infty, \infty)$ except at $x = 0$

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{\frac{-h}{1+h}-0}{-h} = 1$$

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\frac{h}{1+h}-0}{h} = 1$$

∵ $Lf'(0) = Rf'(0) \Rightarrow f$ is differentiable at $x = 0$

Thus, f is differentiable in $(-\infty, \infty)$.

38. **(a,b,d)** If $-1 \le x \le 1$, then $0 \le x \sin \pi x \le 1/2$

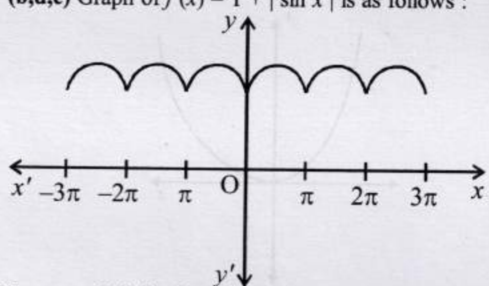
∴ $f(x) = [x \sin \pi x] = 0$

Also $f(x) = [x \sin \pi x] = -1$, when $1 < x < 1 + h$

Thus $f(x)$ is constant and equal to 0 in the closed interval $[-1, 1]$ and so $f(x)$ is continuous and differentiable in the open interval $(-1, 1)$.

At $x = 1$, $f(x)$ is clearly discontinuous, since $f(1-0) = 0$, $f(1+0) = -1$ and $f(x)$ is non-differentiable at $x = 1$.

39. **(b,d,e)** Graph of $f(x) = 1 + |\sin x|$ is as follows :



From graph it is clear that function is continuous every where but not differentiable at integral multiples of π because at these points curve has sharp turnings.

40. **(b)** $f(x) = x(\sqrt{x} - \sqrt{x+1})$

$$Rf'(0) = \lim_{h\to 0}\frac{(0-h)[\sqrt{0-h}-\sqrt{0-h+1}]-0}{-h}$$

$= \lim_{h\to 0}[\sqrt{-h} - \sqrt{-h+1}] = 0 - \sqrt{1} = -1$

$$Rf'(0) = \lim_{h\to 0}\frac{(0+h)[\sqrt{0+h}-\sqrt{0+h+1}]-0}{h}$$

$= \lim_{h\to 0}\sqrt{h} - \sqrt{h+1} = -1$

Since $Lf'(0) = Rf'(0)$

∴ f is differentiable at $x = 0$.

41. **(a, b, d)** Given : $x + |y| = 2y$

If $y < 0$ then $x - y = 2y$

$\Rightarrow y = x/3 \Rightarrow x < 0$

If $y = 0$ then $x = 0$. If $y > 0$ then $x + y = 2y$

$\Rightarrow y = x \Rightarrow x > 0$

∴ $f(x) = y = \begin{cases} x/3, & x<0 \\ x, & x \ge 0 \end{cases}$

Continuity at $x = 0$

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}(-h/3) = 0$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} h = 0$

$f(0) = 0$

∵ LHL = RHL = $f(0)$

∴ $f(x)$ is continuous at $x = 0$

Differentiability at $x = 0$

$Lf' = 1/3$; $Rf' = 1$

As $Lf' \ne Rf' \Rightarrow f(x)$ is not differentiable at $x = 0$

But for $x < 0$, $\dfrac{dy}{dx} = \dfrac{1}{3}$

42. **(c)** Given that $f(x) = \begin{cases} x|x|\sin\frac{1}{x} & ; & x \ne 0 \\ 0 & ; & x = 0 \end{cases}$

and $g(x) = \begin{cases} 1-2x & ; & 0 \le x \le \frac{1}{2} \\ 0 & ; & \text{otherwise} \end{cases}$

$\therefore g\left(\frac{1}{2}-x\right) = \begin{cases} 2x & ; & 0 \le \frac{1}{2}-x \le \frac{1}{2} \\ 0 & ; & \text{otherwise} \end{cases}$

$= \begin{cases} 2x & ; & 0 \le x \le \frac{1}{2} \\ 0 & ; & \text{otherwise} \end{cases}$

Now, $g(x) + g\left(\frac{1}{2}-x\right) = \begin{cases} 1 & ; & 0 \le x \le \frac{1}{2} \\ 0 & ; & \text{otherwise} \end{cases}$

(P) Let $a = 0, b = 1, c = 0, d = 0$

$\therefore h(x) = g(x) + g\left(\frac{1}{2}-x\right) = \begin{cases} 1 & ; & 0 \le x \le \frac{1}{2} \\ 0 & ; & \text{otherwise} \end{cases}$

Hence range of $h(x)$ is $\{0, 1\}$

(Q) Let $a = 1, b = 0, c = 0, d = 0$

$h(x) = f(x) = \begin{cases} x|x|\sin\frac{1}{x} & ; & x \ne 0 \\ 0 & ; & x = 0 \end{cases}$

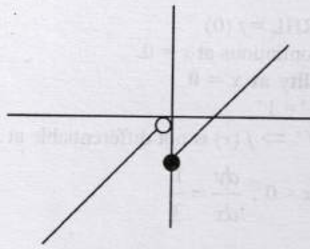
$$\text{RHD} = \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x} = 0$$

$$\text{LHD} = \lim_{x\to 0}\frac{-x^2\sin\frac{1}{x}-0}{x} = 0$$

Hence $h(x)$ is differentiable on R

(R) Let $a = 0, b = 0, c = 1, d = 0$

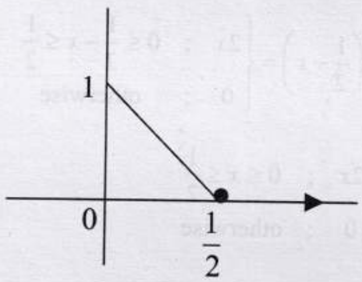
$$h(x) = x - g(x) = \begin{cases} 3x-1 \;; & 0 \le x \le \frac{1}{2} \\ 0 \;; & \text{otherwise} \end{cases}$$



$\therefore h(x)$ is ONTO

(S) Let $a = 0, b = 0, c = 0, d = 1$

$$h(x) = g(x) = \begin{cases} 1-2x \;; & 0 \le x \le \frac{1}{2} \\ 0 \;; & \text{otherwise} \end{cases}$$



Range of $h(x)$ is $[0, 1]$

43. **(d)** (i) $f'_1(0) = \lim_{h\to 0}\left[\frac{\sin\sqrt{1-e^{-h^2}} - 0}{h}\right]$

$$= \lim_{h\to 0}\left[\frac{\sin\sqrt{1-e^{-h^2}}}{\sqrt{1-e^{-h^2}}} \times \frac{\sin\sqrt{1-e^{-h^2}}}{h^2} \times \frac{|h|}{h}\right]$$

$$= \lim_{h\to 0}\left[1 \times 1 \times \frac{|h|}{h}\right] = \lim_{h\to 0}\frac{|h|}{h} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

which does not exit.

$\therefore$ for (P), (2) is correct.

(ii) $\lim_{x\to 0} f_2(x) = \lim_{x\to 0}\left[\frac{|\sin x|}{\tan^{-1} x}\right]$

$$\lim_{x\to 0}\left[\frac{|\sin x|}{|x|} \times \frac{x}{\tan^{-1} x} \times \frac{|x|}{x}\right]$$

$$\lim_{x\to 0}\left[1 \times 1 \times \frac{|x|}{x}\right] = \lim_{x\to 0}\frac{|x|}{x} \qquad \left[\because \lim_{x\to\infty}\frac{x}{\tan^{-1} x} = 1\right]$$

which does not exist, so for Q, (1) is correct.

(iii) $\lim_{x\to 0} f_3(x) = \lim_{x\to 0}[\sin(\log_e(x+2))]$

if $x \to 0 \Rightarrow (x+2) \to 2 \Rightarrow \log_e(x+2) \to \log_e 2 < 1$

$\Rightarrow 0 < \lim_{x\to 0}\sin(\log_e(x+2)) < \sin 1$

$\Rightarrow \lim_{x\to 0}[\sin(\log_e(x+2))] = 0$

$f_3(x) = 0 \quad \forall x \in [-1, e^{\pi/2} - 2)$

$\Rightarrow \quad f'_3(x) = 0 \quad \forall x \in (-1, e^{\pi/2} - 2)$

$\Rightarrow \quad f''_3(x) = 0 \quad \forall x \in (-1, e^{\pi/2} - 2)$

$\therefore$ for (R), (4) is correct.

(iv) $\lim_{x\to 0} f_4(x) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0} x^2\left(\sin\frac{1}{x}\right) = 0$

$$f'_4(0) = \lim_{x\to 0}\frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{x\to 0} h\sin\left(\frac{1}{4}\right) = 0$$

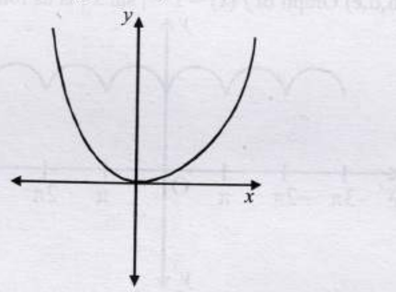
$$f'_4(x) = -\cos\frac{1}{x} + 2x\sin\frac{1}{x}, \; x \ne 0$$

$$\lim_{x\to 0} f'_4(x) = \lim_{x\to 0}\left[-\cos\frac{1}{x} + 2x\sin\frac{1}{x}\right] = -\lim_{x\to 0}\cos\frac{1}{x}$$

which does not exist

So for (S), (3) is correct.

44. **(d)** $P(1)$: $f_4(x) = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \ge 0 \end{cases}$

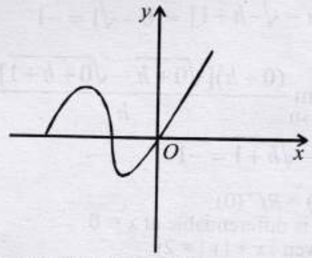


Range of $f_4 = [0, \infty)$

$\therefore f_4$ is onto.

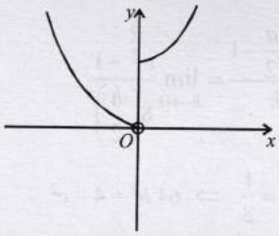
From graph f_4 is not one one.

$Q(3)$: $f_3(x) = \begin{cases} \sin x, & x < 0 \\ x, & x \ge 0 \end{cases}$



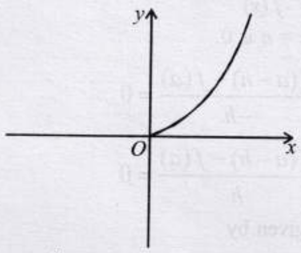
From graph f is differentiable but not one one.

$R(2)$: $f_2 0 f_1(x) = \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \ge 0 \end{cases}$



From graph $f_2 0 f_1$ is neither continuous nor one one.

$S(4)$: $f_2(x) = x^2, x \in [0, \infty)$

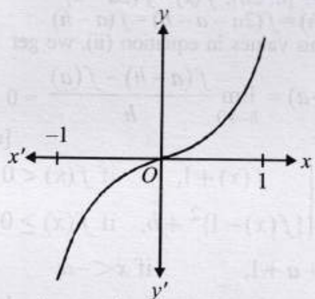


It is continuous and one one.

45. A → (p, q, r); B → (p, s); C → (s, r); D → (p, q)

(A) $y = x|x| = \begin{cases} -x^2, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0 \end{cases}$

Graph is as follows :



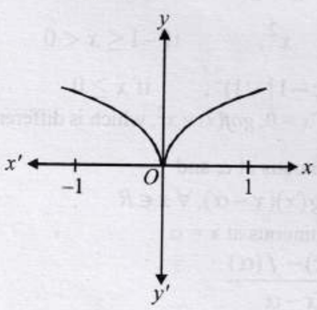
From graph, $y = x|x|$ is continuous in (– 1, 1) (p)
differentiable in (– 1, 1) (q)
and strictly increasing in (–1, 1). (r)

(B) $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases}$

⇒ $y^2 = -x, x < 0$ [where y can take only + ve values]

and $y^2 = x, x \geq 0$

∴ Graph is as follows :

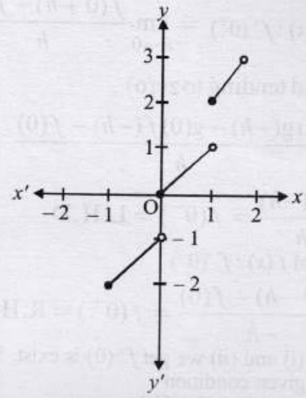


From graph $y = \sqrt{|x|}$ is continuous in (– 1, 1) (p)

and not differentiable at $x = 0$ (s)

(C) $y = x + [x] = \begin{cases} x-1, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ x+1, & 1 \leq x < 2 \end{cases}$

Graph of $y = x + [x]$ is as follows :

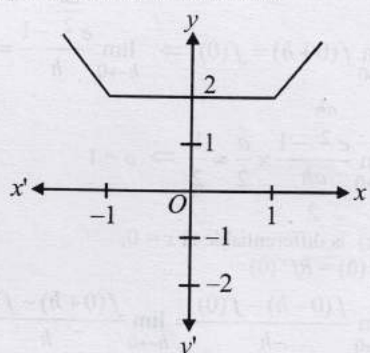


From graph, $y = x + [x]$ is neither continuous, nor differentiable at $x = 0$ and hence in (– 1, 1). (s)

Also it is strictly increasing in (– 1, 1) (r)

(D) $y = |x-1| + |x+1| = \begin{cases} -2x, & x < -1 \\ 2, & -1 \leq x < 1 \\ 2x, & x \geq 1 \end{cases}$

Graph of function is as follows :



From graph, $y = f(x)$ is continuous (p) and differentiable (q) in (– 1, 1) but not strictly increasing in (– 1, 1).

46. **A → (p); B → (r)**

(A) $\sin(\pi[x]) = 0, \forall x \in R$

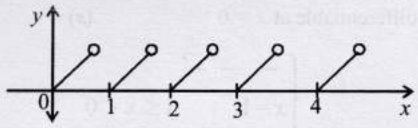
∴ Differentiable everywhere.

∴ (A) → (p)

(B) $\sin(\pi(x - [x])) = f(x)$

We know that $x - [x] = \begin{cases} x, \text{if } 0 \leq x < 1 \\ x-1, \text{if } 1 \leq x < 2 \\ x-2, \text{if } 2 \leq x < 3 \end{cases}$

It's graph is, as shown in figure, which is discontinuous at $\forall x \in z$.



Clearly $x - [x]$ and hence $\sin(\pi(x - [x]))$ is not differentiable $\forall\ x \in z$

(B) → r

47. Given : $f(x-y) = f(x).g(y) - f(y).g(x)$

Put y = x and we get $f(0) = 0$

put $y = 0$ and we get $g(0) = 1$

R.H.D. of $f(x) : f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

($h \in R^+$ and tending to zero)

$= \lim_{h\to 0} \frac{f(0)g(-h) - g(0)f(-h) - f(0)}{h}$

$= \lim_{h\to 0} \frac{-f(-h)}{h} = f(0^-) = \text{L.H.D}$ ...(i)

and L.H.D. of $f(x) : f'(0^-)$

$= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = f(0^+) = \text{R.H.D}$ ...(ii)

Hence from (i) and (ii) we get $f'(0)$ is exist. So it is finite.

Put $y = x$ in given condition

$g(x-y) = g(x).g(y) + f(x).f(y)$

$\Rightarrow g(0) = g^2(x) + f^2(x)$

$\Rightarrow g^2(x) + f^2(x) = 1 \Rightarrow g^2(x) = 1 - f^2(x)$

On diff. w.r.t.x, we get

$2g(x).g'(x) + 2f(x).f'(x) = 0 \Rightarrow g'(0) = 0$

[**Note :** g is differentiable at zero because f is diff. at 0 and $g^2(x) = 1 - f^2(x)$]

48. Given : $f(x)$ is differentiable at $x = 0$.

$\therefore$ $f(x)$ will also be continuous at $x = 0$

$\Rightarrow \lim_{h\to 0} f(0+h) = f(0) \Rightarrow \lim_{h\to 0} \frac{e^{\frac{ah}{2}} - 1}{h} = \frac{1}{2}$

$\Rightarrow \lim_{h\to 0} \frac{e^{\frac{ah}{2}} - 1}{\frac{ah}{2}} \times \frac{a}{2} = \frac{1}{2} \Rightarrow a = 1$

Since $f(x)$ is differentiable at $x = 0$,

$Lf'(0) = Rf'(0)$

$\therefore \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$\Rightarrow \lim_{h\to 0} \frac{b\sin^{-1}\left(\frac{c-h}{2}\right) - \frac{1}{2}}{-h} = \lim_{h\to 0} \frac{e^{\frac{h}{2}} - 1 - \frac{1}{2}}{h}$

$= \lim_{h\to 0} \frac{2e^{\frac{h}{2}} - 2 - h}{2h^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$\Rightarrow \lim_{h\to 0} \frac{\frac{b}{\sqrt{1-\left(\frac{c-h}{2}\right)^2}\cdot\left(-\frac{1}{2}\right)}}{-1}$ [using LH rule]

$= \lim_{h\to 0} \frac{2.e^{\frac{h}{2}}.\frac{a}{2} - 1}{4h} = \lim_{h\to 0} \frac{e^{\frac{h}{2}} - 1}{8\left(\frac{h}{2}\right)}$ $[\because a = 1]$

$\Rightarrow \frac{b}{\sqrt{1-\frac{c^2}{4}}} = \frac{1}{8} \Rightarrow 64\,b^2 = 4 - c^2$

49. Given that $f : [-2a, 2a] \to R$ is an odd function.

$\therefore$ $f(-x) = -f(x)$

Lf' at $x = a$ is 0.

$\Rightarrow \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = 0$

$\Rightarrow \lim_{h\to 0} \frac{f(a-h)-f(a)}{h} = 0$ ...(i)

$Lf'(-a)$ is given by

$\lim_{h\to 0} \frac{f(-a-h)-f(-a)}{-h} = \lim_{h\to 0} \frac{-f(a+h)+f(a)}{-h}$ $[\because f(-x) = -f(x)]$

$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ ...(ii)

Now, for $x \in [a, 2a]$, $f(x) = f(2a-x)$

$\therefore$ $f(a+h) = f(2a-a-h) = f(a-h)$

Substituting this values in equation (ii), we get

$Lf'(-a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{h} = 0$

[using equation (i)]

50. $gof(x) = \begin{cases} f(x)+1, & \text{if } f(x) < 0 \\ \{f(x)-1\}^2 + b, & \text{if } f(x) \ge 0 \end{cases}$

$= \begin{cases} x+a+1, & \text{if } x < -a \\ (x+a-1)^2 + b & \text{if } -a \le x < 0 \\ (|x-1|-1)^2 + b, & \text{if } x \ge 0 \end{cases}$

As $gof(x)$ is continuous at $x = -a$

$gof(-a) = gof(-a^+) = gof(-a^-)$

$\Rightarrow 1 + b = 1 + b = 1 \Rightarrow b = 0$

Also, gof (x) is continuous at $x = 0$

$\Rightarrow gof(0) = gof(0^+) = gof(0^-)$

$\Rightarrow b = b = (a-1)^2 + b \Rightarrow a = 1$

Hence, $gof(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x^2, & \text{if } -1 \le x < 0 \\ (|x-1|-1)^2, & \text{if } x \ge 0 \end{cases}$

In the neighbourhood of $x = 0$, $gof(x) = x^2$, which is differentiable at $x = 0$.

51. (I) Given : g is continuous at α and

$f(x) - f(\alpha) = g(x)(x-\alpha), \forall x \in R$

$\Rightarrow$ Since g is continuous at $x = \alpha$

and $g(x) = \frac{f(x)-f(\alpha)}{x-\alpha}$

$\therefore \lim_{x\to\alpha} g(x) = g(\alpha)$

$\Rightarrow \lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha}=g(\alpha) \Rightarrow f'(\alpha)=g(\alpha)$

$\Rightarrow$ $f'(\alpha)$ exists and is equal to $g(\alpha)$.

(II) $f(x)$ is differentiable at $x=\alpha$ (Given)

$\therefore \lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha}=f'(\alpha)$ exists and is finite.

Let us define, $g(x)=\begin{cases}\frac{f(x)-f(\alpha)}{x-\alpha}, & x\neq\alpha\\ f'(\alpha), & x=\alpha\end{cases}$

Then, $f(x)-f(\alpha)=(x-\alpha)g(x)$, $\forall\ x\neq\alpha$.

Now for continuity of $g(x)$ at $x=\alpha$

$$\lim_{x\to\alpha}g(x)=\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha}=f'(\alpha)=g(\alpha)$$

$\therefore$ g is continuous at $x=\alpha$.

52. Given : $f(x)=\begin{cases}1-x, x<1\\(1-x)(2-x), 1\le x\le 2\\3-x, x>2\end{cases}$

It is clear that the function f is continuous and differentiable at all points except possibility at $x=1$ and $x=2$.

Continuity at $x=1$:

L.H.L. $=\lim_{h\to0}[1-(1-h)]=\lim_{h\to0}h=0$

R.H.L. $=\lim_{h\to0}[1-(1+h)][2-(1+h)]=0$

and $f(1)=0$, $\therefore$ L.H.L. = R.H.L. $=f(1)=0$

Hence, f is continuous at $x=1$

Differentiability at $x=1$.

$$Lf'(1)=\lim_{h\to0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to0}\frac{(1-(1-h)-0}{-h}=-1$$

and $$Rf'(1)=\lim_{h\to0}\frac{f((1+h)-f(1)}{h}$$

$$=\lim_{h\to0}\frac{\{1-(1-h)\}\{2-(1-h)\}-0}{h}=\lim_{h\to0}\frac{-h(1-h)}{h}=-1$$

$\because$ $Lf'(1)=Rf'(1)$

$\therefore$ f is differentiable at $x=1$

Continuity at $x=2$:

L.H.L. $=\lim_{h\to0}[1-(2-h)][2-(2-h)]=0$

and R.H.L. $=\lim_{h\to0}[3-(2+h)]=1$

$\because$ L.H.L. $\neq$ R.H.L., $\therefore$ f is not continuous at $x=2$ and hence f cannot be differentiable at $x=2$.

$\therefore$ f is continuous and differentiable at all points except at $x=2$.

53. Given : $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ ...(i)

On putting $y=0$ in (i), we get

$f\left(\frac{x}{2}\right)=\frac{1}{2}[f(x)+1]$ $[\because f(0)=1]$

$\therefore f(x)=2f\left(\frac{x}{2}\right)-1$ ...(ii)

Now, $f'(x)=\underset{h\to0}{Lt}\frac{f(x+h)-f(x)}{h}$

$=\underset{h\to0}{Lt}\frac{1}{h}\left[\frac{f(2x)+f(2h)}{2}-f(x)\right]$, [using (i)]

$=\underset{h\to0}{Lt}\frac{1}{h}\left[\frac{(2f(x)-1)+(2f(h)-1)}{2}-f(x)\right]$, [using (ii)]

$=\underset{h\to0}{Lt}\frac{1}{h}[f(h)-1]=\underset{h\to0}{Lt}\frac{f(h)-f(0)}{h}$ $[\because f(0)=1]$

$=f'(0)=-1$ $[\because f'(0)=-1]$

On integrating both sides w.r.t. x, we get

$f(x)=-x+c$. On putting $x=0$, we get

$f(0)=c=1$ $[\because f(0)=0]$ $\therefore f(x)=1-x$

$\Rightarrow f(2)=1-2=-1$

54. $f(x+y)=f(x)f(y)$, $\forall\ x,y\in R$

Hence, for $x=y=0$, $f(0+0)=f(0)f(0)$

$\Rightarrow f(0)=[f(0)]^2 \Rightarrow f(0)=1$ $[\because f(x)\neq0$, for any $x]$

Again $f'(0)=2$

$$\Rightarrow \lim_{h\to0}\frac{f(0+h)-f(0)}{h}=2 \Rightarrow \lim_{h\to0}\frac{f(0)f(h)-f(0)}{h}=2$$

$$\Rightarrow \lim_{h\to0}\frac{f(0)[f(h)-1]}{h}=2$$

$$\Rightarrow \lim_{h\to0}\frac{f(h)-1}{h}=2 \quad ...(i) \quad [\because f(0)=1]$$

Now, $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to0}\frac{f(x)f(h)-f(x)}{h}=\lim_{h\to0}f(x)\left(\frac{f(h)-1}{h}\right)$

$=f(x)\lim_{h\to0}\left[\frac{f(h)-1}{h}\right] \Rightarrow f'(x)=f(x).\,2$ [using eq. (i)]

Also, $\frac{f'(x)}{f(x)}=2$

On integrating both sides with respect to x, we get

$\log|f(x)|=2x+c$

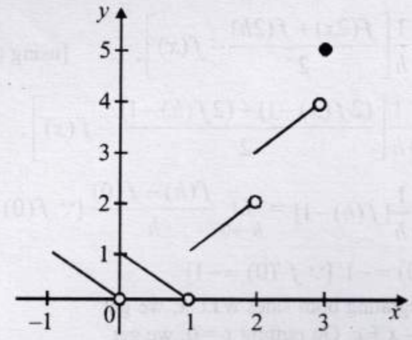
At $x=0$, $\log f(0)=c \Rightarrow c=\log 1=0$

$\Rightarrow \log|f(x)|=2x \Rightarrow f(x)=e^{2x}$

55. Given : $y=[x]+|1-x|$, $-1\le x\le 3$

$$\Rightarrow y=\begin{cases}-1+1-x & ,-1\le x<0\\0+1-x & ,0\le x<1\\1-1+x & ,1\le x<2\\2-1+x & ,2\le x<3\\3-1+x & ,x=3\end{cases}$$

$$\Rightarrow y=\begin{cases}-x, & -1\le x<0\\1-x, & 0\le x<1\\x, & 1\le x<2\\1+x, & 2\le x<3\\2+x, & x=3\end{cases}$$



From graph we can say that given functions is not differentiable at $x = 0, 1, 2, 3$.

56. Given : $f(x)$ is a function satisfying

$f(-x) = f(x), \forall x \in R$

Also $f'(0)$ exists

$\Rightarrow f'(0) = Rf'(0) = Lf'(0)$

Now, $Rf'(0) = Lf'(0)$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{-h}$$

$$\Rightarrow 2\lim_{h\to 0}\frac{f(h)-f(0)}{h} = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)-f(0)}{h} = 0 \Rightarrow f'(0) = 0$$

57. Given $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x-1, & 0 < x \le 2 \end{cases}$

and $g(x) = f(|x|) + |f(x)|$

Here $g(x)$ involves $|x|$ and $|x-1|$ and $|-1| = 1$

Therefore, we should divide the given interval [–2, 2] into the following intervals.

I_1	I_2	I_3
[– 2, 0)	[0, 1)	[1, 2]
$x = -$ve	+ ve	+ ve
$\|x\| = -x$	x	x
$f(x) = -1$	$x-1$	$x-1$
$f(\|x\|) = -1$	$= x-1$	$= x-1$
$\|f(x)\| = \|-1\|$	$\|x-1\|$	$\|x-1\|$
$= 1$	$= -(x-1)$	$= x-1$

$\therefore$ Using above, we get

$g(x) = f|x| + |f(x)|$

$$\Rightarrow g(x) = \begin{cases} -1+1 = 0 \text{ in } I_1 \\ x-1-(x-1) = 0 \text{ in } I_2 \\ x-1+x-1 = 2(x-1) \text{ in } I_3 \end{cases}$$

Hence, g (x) is defined as follows :

$$g(x) = \begin{cases} 0, & -2 \le x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$$

$Lg'(1) = 0; Rg'(1) = 2$

$\therefore$ $g(x)$ is not differentiable at $x = 1$.

58. Here, $f(x) = x^3 - x^2 + x + 1$

$\Rightarrow f'(x) = 3x^2 - 2x + 1$ which is positive $\forall x \in R$

Hence, $f(x)$ is strictly increasing in (0, 2)

$$g(x) = \begin{cases} \max\{f(t)\}, & 0 \le t \le x, 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$$

As $f(x)$ is increasing function

So, max $\{f(t)\}$, $0 \le t \le x, 0 \le x \le 1 = f(x)$

$$\therefore \quad g(x) = \begin{cases} x^3 - x^2 + x + 1 & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$$

Clearly $g(1) = g(1^-) = g(1^+) = 2$

Hence, $g(x)$ is continuous for all $x \in [0, 2]$

$$\text{Also, } g'(x) = \begin{cases} 3x^2 - 2x + 1 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

At $x = 1$, L.H.D. = 2, but R.H.D. = –1

Thus $g(x)$ is not differentiable of $x = 1$

59. Given : $f(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x < 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & 1 \le x \le 2 \end{cases}$

Clearly $f(x)$ may or may not be continuous at $x = 1$ but it is continuous everywhere on [0, 2] except at $x = 1$

$\Rightarrow$ At $x = 1$, $Lf' = \dfrac{2}{2} \times 1 = 1$; $Rf' = 4 \times 1 - 3 = 1$

$\Rightarrow$ f is differentiable and hence continuous at $x = 1$

$\therefore$ $f(x)$ is continuous on [0, 2]

$$f'(x) = \begin{cases} x, & 0 \le x < 1 \\ 4x-3, & 1 \le x \le 2 \end{cases}$$

At $x = 1$,

$$\lim_{x\to 1^-} f'(x) = \lim_{h\to 0} f'(1-h) = \lim_{h\to 0}(1-h) = 1$$

$$\lim_{x\to 1^+} f'(x) = \lim_{h\to 0} f'(1+h) = \lim_{h\to 0} 4(1+h) - 3 = 1$$

$$f'(1) = 4 - 3 = 1$$

Thus $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x) = f'(1)$

$\therefore$ f' is continuous at $x = 1$

Hence, f' is continuous on [0, 2]

$$f''(x) = \begin{cases} 1 & , 0 \le x < 1 \\ 4 & , 1 \le x \le 2 \end{cases}$$

Clearly $f''(x)$ is discontinuous at $x = 1$,

$\therefore$ $f''(x)$ is discontinuous on [0, 2].

60. Given : $f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5} & , x \ne 1 \\ -\dfrac{1}{3} & , x = 1 \end{cases}$

$$\therefore \quad f'(x)|_{x=1} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\dfrac{1+h-1}{2(1+h)^2 - 7(1+h) + 5} + \dfrac{1}{3}}{h}\right] = \lim_{h\to 0}\frac{\dfrac{h}{2h^2 - 3h} + \dfrac{1}{3}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2h-3}+\frac{1}{3}}{h} = \lim_{h\to 0}\frac{2}{3(2h-3)} = -2/9$$

Topic-3: Chain Rule of Differentiation, Differentiation of Explicit & Implicit Functions, Parametric & Composite Functions, Logarithmic & Exponential Functions, Inverse Functions, Differentiation by Trigonometric Substitution

1. **(a)** Given : $\log(x+y) = 2xy$

Clearly, when $x = 0$ then $y = 1$

On differentiating w.r.t. x, we get

$$\frac{1}{x+y}\left[1+\frac{dy}{dx}\right] = 2y + \frac{2xdy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{x+y}-2y}{2x-\frac{1}{x+y}} \Rightarrow y'(0) = \frac{1-2}{0-1} = 1$$

[$\because$ when $x = 0$, then $y = 1$]

2. **(a)** $y = (\sin x)^{\tan x} \Rightarrow \log y = \tan x.\log\sin x$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x\log\sin x + \tan x.\frac{1}{\sin x}.\cos x$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}[1+\sec^2 x\log\sin x]$$

3. **(a)** $f(x) = e^{-x}$ is one such function.

Here $f(0) = 1, f'(0) = -1, f(x) > 0, \forall x$.

$\therefore \quad f''(x) > 0 \;\forall x$

4. **(1)** $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

$$= \sin\left[\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta+\cos 2\theta}}\right)\right]\left[\because \tan^{-1}\frac{x}{y} = \sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right]$$

$$= \sin\left[\sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)\right] = \tan\theta$$

$$\Rightarrow \frac{df(\theta)}{d\tan\theta} = 1.$$

5. **(2)** Given : $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$

therefore we should have $gof(x) = x$

$\therefore \quad g(f(x)) = x \Rightarrow g(x^3+e^{x/2}) = x$

On differentiating both sides w.r.t. x, we get

$$g'(x^3+e^{x/2}).\left(3x^2+e^{x/2}\frac{1}{2}\right) = 1$$

$$\Rightarrow g'(x^3+e^{x/2}) = \frac{1}{3x^2+e^{x/2}.\frac{1}{2}}$$

For $x = 0$, we get $g'(1) = 2$

6. Given : $xe^{xy} = y + \sin^2 x$

Differentiating both sides w. r.to x, we get

$$e^{xy}.1 + xe^{xy}\left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx} + 2\sin x\cos x$$

On putting $x = 0$, we get $1 + 0 = \frac{dy}{dx} + 0 \Rightarrow \frac{dy}{dx} = 1$

7. $f(x) = |x-2|$

$\Rightarrow g(x) = f(f(x)) = |f(x)-2|$ for $x > 20$

$= ||x-2|-2| = |x-2-2|$ for $x > 20$

$= |x-4| = x-4$ for $x > 20$

$\therefore \quad g'(x) = 1 \Rightarrow g'(x) = -4$

8. Let $u = \sec^{-1}\left(\frac{1}{2x^2-1}\right) \Rightarrow u = \cos^{-1}(2x^2-1) = 2\cos^{-1} x$

and $v = \sqrt{1-x^2}$

$$\therefore \quad \frac{du}{dx} = \frac{-2}{\sqrt{1-x^2}} \text{ and } \frac{dv}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-x}{\sqrt{1-x^2}}} = \frac{2}{x} \Rightarrow \left.\frac{du}{dv}\right|_{x=\frac{1}{2}} = 4$$

9. Given that, $f(x) = \log_x(ln\,x) = \frac{\log_e(\log_e x)}{(\log_e x)}$

$$f'(x) = \frac{\frac{1}{\log_e x}\times\frac{1}{x}\times\log_e x - \frac{1}{x}\log_e(\log_e x)}{(\log_e x)^2}$$

$$= \frac{\frac{1}{x}[1-\log_e(\log_e x)]}{(\log_x x)^2}$$

$$f'(e) = \frac{\frac{1}{e}[1-\log_e(\log_e e)]}{(\log_e e)^2} = \frac{\frac{1}{e}[1-\log_e 1]}{(1)^2} = \frac{1}{e}(1-0) = \frac{1}{e}.$$

10. Given that $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$ ... (i)

where $f_r(x), g_r(x), h_r(x), r = 1, 2, 3$, are polynominals in x and hence differentiable and

$f_r(a) = g_r(a) = h_r(a), r = 1, 2, 3$ ... (ii)

On differentiating equation (i) with respect to x, we get

$$F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

$$\Rightarrow F'(a) = \begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$$

$F'(a) = D_1 + D_2 + D_3$

Using equation (ii) and the property of determinants that $D = 0$, if two rows in D are identical, we get $D_1 = D_2 = D_3 = 0$

$\therefore \quad F'(a) = 0.$

11. Given: $y = f\left(\frac{2x-1}{x^2+1}\right); f'(x) = \sin x^2$

$\therefore \frac{dy}{dx} = f'\left(\frac{2x-1}{x^2+1}\right) \cdot \frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$

$= \left[\sin\left(\frac{2x-1}{x^2+1}\right)^2\right]\left[\frac{2(x^2+1)-2x(2x-1)}{(x^2+1)^2}\right]$

$= \frac{2+2x-2x^2}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$

12. **(True)** Consider $g(x) = \frac{f(x)+f(-x)}{2}$, which is an even function

$\therefore g'(x) = \frac{f'(x)-f'(-x)}{2} = h(x)$, let

Now $h(-x) = \frac{f'(-x)-f'(x)}{2} = -h(x)$,

$\therefore h$ is an odd function.

Hence, derivative of an even function is an odd function.

13. **(d)** $f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$

$= \sum_{j=1}^{n} \tan^{-1}\left[\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right]$

$= \sum_{j=1}^{n}[\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$

$\Rightarrow f_n(x) = \tan^{-1}(x+n) - \tan^{-1}(x)$

$= \tan^{-1}\left(\frac{n}{1+x(n+x)}\right) \Rightarrow f_n'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$

and $f_n(0) = \tan^{-1}(n)$, $\therefore \tan^2(\tan^{-1}n) = n^2$

Here $x = 0$ is not in the given domain, i.e., $x \in (0, \infty)$.

$\therefore$ Options (a) & (b) are not correct options.

(c) $\lim_{x\to\infty} \tan(f_n(x)) = \lim_{x\to\infty}\left(\frac{n}{1+x(n+x)}\right) = 0$

(d) $\lim_{x\to\infty} \sec^2(f_n(x)) = \lim_{n\to\infty} 1+\tan^2(f_n(x))$

$= 1 + \lim_{x\to\infty} \tan^2(f_n(x)) = 1$

14. **(a, b, d)**

(a) f(x) being twice differentiable, it is continuous but can't be constant throughout the domain.

Hence we can find $x \in (r, s)$ such that f(x) is one one.

$\therefore$ (a) is true.

(b) By Lagrange's Mean Value theorem for f(x) in [–4, 0], there exists

$x_0 \in (-4, 0)$ such that $f'(x_0) = \frac{f(0)-f(-4)}{0-(-4)}$

$\Rightarrow |f'(x_0)| = \left|\frac{f(0)-f(-4)}{4}\right|$

$\because -2 \le f(x) \le 2$, $\therefore -4 \le f(0)-f(-4) \le 4$

$\Rightarrow |f'(x_0)| \le 1$, $\therefore$ (b) is true.

(c) If we consider $f(x) = \sin(\sqrt{85}x)$ then f (x) satisfies the given condition $[f(0)]^2 + [f'(0)]^2 = 1$

But $\lim_{x\to\infty}(\sin\sqrt{85}x)$ does not exist

$\therefore$ (c) is false.

(d) Let us consider $g(x) = [f(x)]^2 + [f'(x)]^2$

By Lagrange's Mean Value theorem $|f'(x)| \le 1$

Also $|f(x_1)| \le 2$ as $f(x) \in [-2, 2]$

$\therefore g(x_1) \le 5$, for $x_1 \in (-4, 0)$

Similarly $g(x_2) \le 5$, for $x_2 \in (0, 4)$

Also $g(0) = 85$

Hence g(x) has maxima in (x_1, x_2) say at α such that

$g'(\alpha) = 0$ and $g(\alpha) \ge 85$

$g'(\alpha) = 0 \Rightarrow 2f(\alpha)f'(\alpha) + 2f'(\alpha)f''(\alpha) = 0$

$\Rightarrow 2f'(\alpha)[f(\alpha) + f''(\alpha)] = 0$

If $f'(\alpha) = 0 \Rightarrow g(\alpha) = [f(\alpha)]^2$ and $[f(\alpha)]^2 \le 4$

$\therefore g(\alpha) \ge 85$ (is not possible.)

$\Rightarrow f(\alpha) + f''(\alpha) = 0$ for $\alpha \in (x_1, x_2) \in (-4, 4)$

Hence, (d) is true.

15. **(b, c)** Given : $f(x) = x^3 + 3x + 2 \Rightarrow f'(x) = 3x^2 + 3$

$\therefore f(0) = 2, f(1) = 6, f(2) = 16, f(3) = 38, f(6) = 236$

Also given $g(f(x)) = x \Rightarrow g(2) = 0, g(6) = 1, g(16) = 2, g(3, 8) = 3, g(236) = 6$

(a) $g(f(x)) = x \Rightarrow g'(f(x)).\ f'(x) = 1$

For g′(2), $f(x) = 2 \Rightarrow x = 0$

On putting x = 0, we get $g'(f(0))\ f'(0) = 1$

$\Rightarrow g'(2) = \frac{1}{3}$

(b) $h(g(g(x))) = x \Rightarrow h'(g(g(x))).\ g'(g(x)).\ g'(x) = 1$

For h′(1), we need $g(g(x)) = 1$

$\Rightarrow g(x) = 6 \Rightarrow x = 236$

On putting x = 236, we get

$h'[g(g(236))] = \frac{1}{g'(g(236)).g'(236)}$

$\Rightarrow h'(g(6)) = \frac{1}{g'(6)\ g'(236)}$

$\Rightarrow h'(1) = \frac{1}{g'(f(1))\ g'(f(6))}$

$= f'(1)\ f'(6) = 6 \times 111 = 666$

(c) $h[g(g(x))] = x$

For h (0), $g(g(x)) = 0 \Rightarrow g(x) = 2 \Rightarrow x = 16$

On putting x = 16, we get

$h(g(g(16))) = 16 \Rightarrow h(0) = 16$

(d) $h[g(g(x))] = x$

For h(g(3)), we need $g(x) = 3 \Rightarrow x = 38$

On putting x = 38, we get

$h[g(g(38))] = 38 \Rightarrow h(g(3)) = 38$

16. **(b)** Given that $f(x) = 2 + \cos x$ which is continuous and differentiable every where.

Also $f'(x) = -\sin x$, $\therefore f'(x) = 0 \Rightarrow x = n\pi$

$\Rightarrow$ There exists $c \in [t, t+\pi]$ for $t \in R$ such that $f'(c) = 0$

$\therefore$ Statement-1 is true.

Also $f(x)$ being periodic of period 2π, statement-2 is true, but statement-2 is not a correct explanation of statement-1.

17. $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1$

$= \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{x}{x-c}$

$= \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \left(\dfrac{b}{x-b}+1\right)\dfrac{x}{x-c}$

$= \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{x^2}{(x-b)(x-c)}$

$= \left(\dfrac{a}{x-a}+1\right)\dfrac{x^2}{(x-b)(x-c)} = \dfrac{x^3}{(x-a)(x-b)(x-c)}$

Taking log on both sides, we get

$\log y = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$

$\Rightarrow \dfrac{y'}{y} = \dfrac{3}{x} - \dfrac{1}{x-a} - \dfrac{1}{x-b} - \dfrac{1}{x-c}$

$= \left(\dfrac{1}{x}-\dfrac{1}{x-a}\right) + \left(\dfrac{1}{x}-\dfrac{1}{x-b}\right) + \left(\dfrac{1}{x}-\dfrac{1}{x-c}\right)$

$= \dfrac{a}{x(a-x)} + \dfrac{b}{x(b-x)} + \dfrac{c}{x(c-x)}$

$= \dfrac{1}{x}\left[\dfrac{a}{a-x} + \dfrac{b}{b-x} + \dfrac{c}{c-x}\right]$

18. $(\sin y)^{\sin\left(\frac{\pi x}{2}\right)} + \dfrac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x \tan[\ln(x+2)] = 0$ ... (i)

Put $x = -1$, in equation (i), we get

$(\sin y)^{\sin\left(-\frac{\pi}{2}\right)} + \dfrac{\sqrt{3}}{2}\sec^{-1}(-2) + 2^{-1}\tan[\ln(-1+2)] = 0$

$\Rightarrow (\sin y)^{-1} + \dfrac{\sqrt{3}}{2}\left(\dfrac{2\pi}{3}\right) + \dfrac{1}{2}\tan 0 = 0$

$\Rightarrow \sin y = -\dfrac{\sqrt{3}}{\pi}$, when $x = -1$ ....(ii)

Let $u = (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}$

Taking ln on both sides; we get

$\ln u = \sin\left(\dfrac{\pi x}{2}\right)\ln\sin y$

Differentiating both sides with respect to x, we get

$\dfrac{1}{u}\dfrac{du}{dx} = \dfrac{\pi}{2}\cos\left(\dfrac{\pi x}{2}\right)\ln\sin y + \cot y\dfrac{dy}{dx}\sin\left(\dfrac{\pi x}{2}\right)$

$\Rightarrow \dfrac{du}{dx} = (\sin y)^{\sin\left(\frac{\pi x}{2}\right)} \times \left[\dfrac{\pi}{2}\cos\left(\dfrac{\pi x}{2}\right)\ln\sin y + \sin\left(\dfrac{\pi x}{2}\right)\cot y\dfrac{dy}{dx}\right]$ ...(iii)

Now differentiating equation (i), we get

$\dfrac{d}{dx}\left[(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\right] + \dfrac{\sqrt{3}}{2}\dfrac{1}{2x\sqrt{4x^2-1}}.2$

$+ 2^x(\ln 2)\tan[(\ln(x+2)] + 2^x\sec^2[\ln(x+2)]\dfrac{1}{x+2} = 0$

$\Rightarrow (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\left[\dfrac{\pi}{2}\cos\left(\dfrac{\pi x}{2}\right)\ln\sin y + \sin\left(\dfrac{\pi x}{2}\right)\cot y\dfrac{dy}{dx}\right]$

$+ \dfrac{\sqrt{3}}{2x\sqrt{4x^2-1}} + 2^x\ln 2\tan(\ln(x+2))$

$+ \dfrac{2^x\sec^2[\ln(x+2)]}{x+2} = 0$ [using (iii)]

At $x = -1$ and $\sin y = -\dfrac{\sqrt{3}}{\pi}$, we get

$\left(-\dfrac{\sqrt{3}}{\pi}\right)^{-1}\left[0-(-1)\sqrt{\dfrac{\pi^2}{3}-1}\left(\dfrac{dy}{dx}\right)_{x=-1}\right]$

$+ \dfrac{\sqrt{3}}{-2\sqrt{3}} + 0 + 2^{-1} = 0$

$\Rightarrow -\dfrac{\pi}{\sqrt{3}\sqrt{3}}\sqrt{\pi^2-3}\left(\dfrac{dy}{dx}\right)_{x=-1} - \dfrac{1}{2} + \dfrac{1}{2} = 0$

$\Rightarrow \left.\dfrac{dy}{dx}\right]_{x=-1} = 0$

19. Given : $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$

$\Rightarrow \dfrac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$

$= \sec\theta\tan\theta + \tan\theta\cos\theta = \tan\theta(\sec\theta + \cos\theta)$

and $\dfrac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta - n\cos^{n-1}\theta(-\sin\theta)$

$= n\tan\theta(\sec^n\theta + \cos^n\theta)$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{n\tan\theta(\sec^n\theta+\cos^n\theta)}{\tan\theta(\sec\theta+\cos\theta)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{n(\sec^n\theta+\cos^n\theta)}{(\sec\theta+\cos\theta)}$ ...(i)

Now $x^2 + 4 = (\sec\theta - \cos\theta)^2 + 4$

$= \sec^2\theta + \cos^2\theta - 2\sec\theta\cos\theta + 4$

$= \sec^2\theta + \cos^2\theta + 2 = (\sec\theta + \cos\theta)^2$ ...(ii)

and $y^2 + 4 = (\sec^n\theta - \cos^n\theta)^2 + 4$

$= \sec^{2n}\theta + \cos^{2n}\theta - 2\sec^n\theta\cos^n\theta + 4$

$= \sec^{2n}\theta + \cos^{2n}\theta + 2 = (\sec^n\theta + \cos^n\theta)^2$ ...(iii)

$= n^2(y^2+4)$

Now $(x^2+4)\left(\dfrac{dy}{dx}\right)^2 = (\sec\theta+\cos\theta)^2 . \dfrac{n^2(\sec^n\theta+\cos^n\theta)^2}{(\sec\theta+\cos\theta)^2}$ [using (i) and (ii)]

$= n^2(\sec^n\theta + \cos^n\theta)^2$

$\Rightarrow (x^2+4)\left(\dfrac{dy}{dx}\right)^2 = n^2(y^2+4)$ [From (iii)]

20. Let $F(x)=\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$ .... (i)

Since α is a repeated root of quadratic equation $f(x) = 0$

$\therefore$ We must have $f(x) = k(x-\alpha)^2$; where k is a non- zero real number.

Put $x = \alpha$ on both sides of equation (i); we get

$$F(\alpha)=\begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}=0$$

[$\because$ R_1 and R_2 are identical]

$\therefore$ $(x-\alpha)$ is a factor of $F(x)$

Differentiating equation (i) w.r. to x, we get

$$F'(x)=\begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$$

Putting $x = \alpha$, we get

$$F'(\alpha)=\begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}=0$$

[$\because$ R_1 and R_3 are identical]

$\Rightarrow$ $(x-\alpha)$ is a factor of $F'(x)$.

$\Rightarrow$ $(x-\alpha)^2$ is a factor of $F(x)$.

$\therefore$ $F(x)$ is divisible by $f(x)$.

21. Given : $y = e^{x\sin x^3} + (\tan x)^x$

Here y is the sum of two functions and in the second function base as well as power are functions of x. Therefore, here we will use logarithmic differentiation.

Let $y = u + v$, where $u = e^{x\sin x^3}$ and $v = (\tan x)^x$

$$\therefore \quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad ...(i)$$

Now, $\dfrac{du}{dx}=e^{x\sin x^3}\cdot\dfrac{d}{dx}(x\sin x^3)$

$= e^{x\sin x^3}\cdot[3x^3 \cdot \cos x^3 + \sin x^3]$

Now $v = (\tan x)^x \Rightarrow \log v = x \log \tan x$

Now, differentiating the both sides with respect to x, then

$$\frac{1}{v}\frac{dv}{dx}=x.\frac{1}{\tan x}.\sec^2 x+1.\log\tan x$$

$$\therefore \quad \frac{dv}{dx}=(\tan x)^x\left[\frac{2x}{\sin 2x}+\log\tan x\right]$$

Now substituting the value of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ in (i), we get

$$\frac{dy}{dx}=e^{x\sin x^3}[\sin x^3+3x^3\cos x^3] +(\tan x)^x\left[\frac{2x}{\sin 2x}+\log\tan x\right]$$

22. Given : $y=\dfrac{5x}{3|1-x|}+\cos^2(2x+1)$

[Clearly y is not defined at $x = 1$]

$$\Rightarrow y=\begin{cases}\dfrac{5x}{3(1-x)}+\cos^2(2x+1), & x<1\\ \dfrac{5x}{3(x-1)}+\cos^2(2x+1), & x>1\end{cases}$$

$$\Rightarrow \frac{dy}{dx}=\begin{cases}\dfrac{5}{3}\left(\dfrac{(1-x)-x(-1)}{(1-x)^2}\right)-2\sin(4x+2), & x<1\\ \dfrac{5}{3}\left(\dfrac{(x-1)-x}{(x-1)^2}\right)-2\sin(4x+2), & x>1\end{cases}$$

$$\text{or} \quad \frac{dy}{dx}=\begin{cases}\dfrac{5}{3}\dfrac{1}{(1-x)^2}-2\sin(4x+2), & x<1\\ -\dfrac{5}{3}\dfrac{1}{(x-1)^2}-2\sin(4x+2), & x>1\end{cases}$$

Topic-4: Differentiation of Infinite Series, Successive Differentiation, nth Derivative of Some Standard Functions, Leibnitz's Theorem, Rolle's Theorem, Lagrange's Mean Value Theorem

1. **(a)** Given : $g(x) = \log f(x) \Rightarrow g(x+1) = \log f(x+1)$

$\Rightarrow$ $g(x+1) = \log x f(x)$ [$\because f(x+1) = x f(x)$]

$\Rightarrow g(x+1) = \log x + \log f(x) \Rightarrow g(x+1) - g(x) = \log(x)$

$$\Rightarrow g'(x+1)-g'(x)=\frac{1}{x} \Rightarrow g''(x+1)-g''(x)=-\frac{1}{x^2}$$

On putting, $x = x - \dfrac{1}{2}$, we get

$$\Rightarrow g''\left(x+\frac{1}{2}\right)-g''\left(x-\frac{1}{2}\right)=-\frac{1}{\left(x-\frac{1}{2}\right)^2}=\frac{-2^2}{(2x-1)^2}$$

On putting $x = 1, 2, 3$ ...., N; we get

$$g''\left(\frac{3}{2}\right)-g''\left(\frac{1}{2}\right)=-\frac{2^2}{1^2} \quad ...(i)$$

$$g''\left(\frac{5}{2}\right)-g''\left(\frac{3}{2}\right)=\frac{-2^2}{3^2} \quad ...(ii)$$

$$g''\left(\frac{7}{2}\right)-g''\left(\frac{5}{2}\right)=\frac{-2^2}{5^2} \quad ...(iii)$$

$$g''\left(N+\frac{1}{2}\right)-g''\left(N-\frac{1}{2}\right)=-\frac{2^2}{(2N-1)^2} \quad ...(N)$$

On adding all the above equations, we get

$$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4\left[1+\frac{1}{3^2}+\frac{1}{5^2}+.....+\frac{1}{(2N-1)^2}\right]$$

2. **(d)** $\frac{d^2x}{d^2y} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right) \times \frac{dx}{dy}$

$$= \left\{\frac{d}{dx}\left[\frac{1}{\left(\frac{dy}{dx}\right)}\right]\right\} \times \frac{1}{\frac{dy}{dx}} = -\frac{1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d^2y}{dx^2} \times \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= -\left(\frac{dy}{dx}\right)^{-3} \frac{d^2y}{dx^2}$$

3. **(d)** Let us consider the function $g(x) = f(x) - x^2$

such that $g(1) = f(1) - 1^2 = 1 - 1 = 0$

$g(2) = f(2) - 2^2 = 4 - 4 = 0$

$g(3) = f(3) - 3^2 = 9 - 9 = 0$

Since $f(x)$ is twice differentiable, therefore we can say $g(x)$ is continuous and differentiable everywhere and

$g(1) = g(2) = g(3) = 0$

$\therefore$ By Rolle's theorem, $g'(c) = 0$ for some $c \in (1, 2)$ and

$g'(d) = 0$ for some $d \in (2, 3)$

Again by Rolle's theorem,

$g''(e) = 0$ for some $e \in (c, d) \Rightarrow e \in (1, 3)$

$\Rightarrow f''(e) - 2 = 0$ or $f''(e) = 2$ for some $x \in (1, 3)$

$\therefore f''(x) = 2$ for some $x \in (1, 3)$

4. **(b)** $x^2 + y^2 = 1 \Rightarrow 2x + 2yy' = 0 \Rightarrow x + yy' = 0$

$\Rightarrow 1 + yy'' + (y')^2 = 0 \Rightarrow yy'' + (y')^2 + 1 = 0$

5. **(b)** Let $f(x) = ax^2 + bx + c$ and $f(x) > 0, \ \forall \ x \in R$

$\therefore a > 0$ and $D < 0$

$\Rightarrow a > 0$ and $b^2 - 4ac < 0$ ..... (i)

Now, $g(x) = f(x) + f'(x) + f''(x)$

$= ax^2 + bx + c + 2ax + b + 2a$

$= ax^2 + (2a + b)x + (2a + b + c)$

Here, $D = (2a + b)^2 - 4a(2a + b + c)$

$= 4a^2 + b^2 + 4ab - 8a^2 - 4ab - 4ac$

$= b^2 - 4a^2 - 4ac = -4a^2 + b^2 - 4ac$

$= (-ve) + (-ve) = -ve$ [using (i)]

Also from (i), $a > 0$

$\therefore g(x) > 0, \ \forall \ x \in R$

6. **(c)** Given : $y^2 = P(x)$, where $P(x)$ is a polynomial of degree 3 and hence thrice differentiable.

$\therefore 2y\frac{dy}{dx} = P'(x)$ ... (i)

Again differentiating with respect to x, we get

$$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$$

$\Rightarrow \frac{[P'(x)]^2}{2y^2} + 2y\frac{d^2y}{dx^2} = P''(x)$ [using (i)]

$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2y^2P''(x) - [P'(x)]^2$

$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2P(x)P''(x) - [P'(x)]^2$ $[\because y^2 = p(x)]$

$\Rightarrow 2y^3\frac{d^2y}{dx^2} = P(x)P''(x) - \frac{1}{2}[P'(x)]^2$

Again on differentiating w.r. to x, we get $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$

$= P'''(x)P(x) + P''(x)P'(x) - P'(x)P''(x) = P'''(x)P(x)$

7. **(5.00)** $g(x) = (x^2 - 1)^2 h(x)$; $h(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$\because f(1) = f(-1) = 0$

$\Rightarrow f(x)$ has two roots $x = 1$ and $x = -1$

$\Rightarrow g(x)$ has atleast 3 roots $x = 1, x = -1$ and $x = \alpha$

Then by Rolle's theorem

$\Rightarrow g'(\alpha) = 0, \alpha \in (-1, 1)$

$g'(1) = g'(-1) = 0 \Rightarrow g'(x) = 0$ has atleast 3 root,

$\Rightarrow g''(x) = 0$ will have at leeast 2 root, say β, γ such that

Then by Rolle's theorem

$-1 < \beta < \alpha < \gamma < 1$

So, $\min(m_{f''}) = 2$ and we find $(m_{f'} + m_{f''}) = 5$

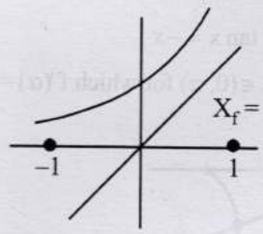
8. **(a, b, c)** Given, S = Set of twice differentiable functions f : $R \to R$

$\frac{d^2f}{dx^2} > 0$ in $(-1, 1)$

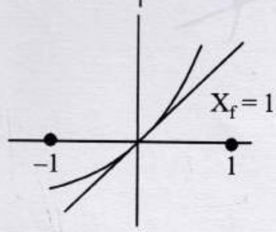
Graph 'f' is Concave upward.

Number of solutions of $f(x) = x \to x_f$

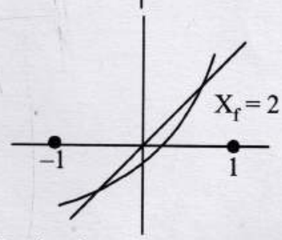
(1)



(2)



(3)



$\Rightarrow$ Graph of $y = f(x)$ can intersect graph of $y = x$ at atmost two points $\Rightarrow 0 \le x_f \le 2$

9. **(b, c, d)** $\lim_{t\to x}\frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2 x$

$\Rightarrow \lim_{t\to x}\frac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$ (using LH Rule)

$\Rightarrow f(x)\cos x - f'(x)\sin x = \sin^2 x$

$\Rightarrow -\left(\frac{f'(x)\sin x - f(x)\cos x}{\sin^2 x}\right) = 1$

$\Rightarrow -d\left(\frac{f(x)}{\sin x}\right) = 1 \Rightarrow \frac{f(x)}{\sin x} = -x + c$

Put $x = \frac{\pi}{6}$

$\therefore \frac{\frac{-\pi}{12}}{\frac{1}{2}} = -\frac{\pi}{6} + c \quad \left[\because f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}\right]$

$\Rightarrow \frac{-\pi}{12} = \frac{-\pi}{12} + c \Rightarrow c = 0 \Rightarrow f(x) = -x\sin x$

(a) $f\left(\frac{\pi}{4}\right) = \frac{-\pi}{4}\frac{1}{\sqrt{2}}$

(b) $f(x) = -x\sin x$

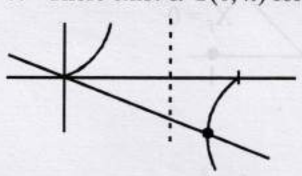
$\because \sin x > x - \frac{x^3}{6} \; \forall x \in (0, \pi)$

$\therefore -x\sin < -x^2 + \frac{x^4}{6} \Rightarrow f(x) < -x^2 + \frac{x^4}{6} \; \forall x \in (0,\pi)$

(c) $f'(x) = -\sin x - x\cos x$

Now $f'(x) = 0 \Rightarrow \tan x = -x$

$\therefore$ There exist $\alpha \in (0, \pi)$ for which $f'(\alpha) = 0$



(d) Here, $f''(x) = -2\cos x + x\sin x$

$\therefore f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} \Rightarrow f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

10. **(a)** We have $f(x) = g(x)\sin x$

$\Rightarrow f'(x) = g'(x)\sin x + g(x)\cos x$

$\Rightarrow f'(0) = g'(0)\times 0 + g(0) = g(0) \quad [\because g'(0) = 0]$

$\therefore$ Statement 2 is correct.

Also $f''(0) = \lim_{x\to 0}\frac{f'(x) - f'(0)}{x}$

$= \lim_{x\to 0}\frac{g(x)\cos x + g'(x)\sin x - g(0)}{x}$

$= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{x} + \lim_{x\to 0}\frac{g'(x)\sin x}{x}$

$= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{x\times\frac{\sin x}{x}} + \lim_{x\to 0} g'(x)$

$= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{\sin x} + g'(0)$

$= \lim_{x\to 0}\left[g(x)\cot(x) - g(0)\operatorname{cosec} x\right] + 0$

$= \lim_{x\to 0}\left[g(x)\cot x - g(0)\operatorname{cosec} x\right]$

$\therefore$ Statement 1 is also true and statement 2 is a correct explanation for statement 1.

11. Given : f is twice differentiable such that

$f''(x) = -f(x)$ and $f'(x) = g(x)$

$h(x) = [f(x)]^2 + [g(x)]^2$

$\Rightarrow h'(x) = 2ff' + 2gg' = 2f(x)\,g(x) + 2g(x)f''(x)$

$[\because g(x) = f'(x) \Rightarrow g'(x) = f''(x)]$

$= 2f(x)\,g(x) + 2g(x)(-f(x)) \quad [\because f''(x) = -f(x)]$

$= 2f(x)\,g(x) - 2f(x)\,g(x) = 0$

$\therefore h'(x) = 0,$ for all $x \Rightarrow h$ is a constant function

$\therefore h(5) = 11 \Rightarrow h(10) = 11.$